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Cohomology of Standard Blowing-Up

M. BRODMANN

Mathematisches Institut der Universitaet Zuerich, Raemistrasse 74, CH-8001 Zuerich, Switzerland

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1. INTRODUCTION

Let $X = \operatorname{Spec}(R)$ be a noetherian affine scheme and let $\mathscr{F}$ be a coherent sheaf over X. Let $p \in X$ be a closed point. We write d for the dimension $\dim_p(X)$ of X at p. We assume that the local cohomology modules

$$(1.1) \qquad H_p^i(\mathscr{F}) = H^i_{\mathfrak{m}_{X,p}}(\mathscr{F}_p)$$

are finitely generated for all $i < d$.

Then X admits a particular class of blowing-up morphisms

$$(1.2) \qquad \pi \colon \tilde{X} \to X,$$

the so-called *standard blowing-up* (with respect to the sheaf $\mathscr{F}$) centered at p. A precise definition of this notion will be given in the next section. Standard blowing-up has been studied by different authors (cf. Brodmann [2, 4–7], Faltings [10], Goto [12, 13], Goto and Shimoda [14], Goto and Yamagishi [15], Schenzel [23], Trung [28]).

It plays a crucial role in the search for certain birational models—the so-called Macaulayfications (cf. [2, 4, 6, 10].

Together with the sheaf $\mathscr{F}$ and the blowing-up morphism π we have an $\mathcal{O}_{\tilde{X}}$-sheaf $\tilde{\mathscr{F}}$, which we call the *Rees modification* of $\mathscr{F}$ with respect to π. A precise definition of this notion will be given in the next section. For the moment we only remark that $\tilde{\mathscr{F}}$ is coherent and that $\tilde{\mathcal{O}}_X = \mathcal{O}_{\tilde{X}}$.

The aim of this paper is to study the modification $\tilde{\mathscr{F}}$. In particular we are interested in the cohomology groups of $\tilde{X}$ with coefficients in this sheaf $\tilde{\mathscr{F}}$. More precisely, our goal is to express the total cohomology modules

$$(1.3) \qquad H^i_*(\tilde{X}, \tilde{\mathscr{F}}) := \bigoplus_{n \in \mathbb{Z}} H^i(\tilde{X}, \tilde{\mathscr{F}}(n)) \qquad (i < d-1)$$

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and the lengths of the modules

$$H^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n)) \qquad (n \in \mathbb{Z}) \tag{1.4}$$

in terms of the modules $H^i_p(\mathscr{F})$ $(1 < i < d)$ and $\Gamma(X - \{p\}, \mathscr{F})$. Thereby twisting of the sheaf $\tilde{\mathscr{F}}$ is understood with respect to the (relative) very ample sheaf $\mathcal{O}_{\tilde{X}}(1)$ induced by the projective morphism π.

The motivation of study the modification $\tilde{\mathscr{F}}$ originates from the Macaulayfications already mentioned above. Indeed, one of the main results of [10] claims that if $X - \{p\}$ is a Cohen–Macaulay (CM) scheme and if $\pi: \tilde{X} \to X$ is a standard blowing-up of X (with respect to $\mathcal{O}_X$) centered at p, then $\tilde{X}$ is a CM scheme. Thereby, if X is of finite type over a field and of pure dimension, we need not care about the finiteness of the local cohomology modules $H^i_p(\mathcal{O}_X)$ for $i < d$. Namely, by Grothendieck's finiteness theorem [18], it follows from the CM-property of $X - \{p\}$ that these modules are of finite length in this case. We even may replace the hypothesis that X is of finite type over a field by the weaker hypothesis that X is a closed subscheme of a regular scheme, or —even weaker—that the formal fibers of X have the CM-property [11].

In the situation of our paper, things behave very similarly. Namely, if the stalk $\mathscr{F}_q$ is a maximal CM-module for all $q \in X - \{p\}$, then $\tilde{\mathscr{F}}_s$ is a maximal CM-module for all $s \in \tilde{X}$. Here again, if the formal fibers of X have the CM-property and if X is of pure dimension, the finiteness of the modules $H^i_p(\mathscr{F})$ $(i < d)$ already follows from the CM-property of the stalks $\mathscr{F}_q$ $(q \in X - \{p\})$. This is a consequence of Faltings' version [11] of Grothendieck's finiteness theorem.

To investigate the Rees modification $\tilde{\mathscr{F}}$ we first study its *exceptional sheaf* $\tilde{\mathscr{F}} \mid E$, e.g., the restriction of $\tilde{\mathscr{F}}$ to the exceptional fiber of the blowing-up π. More precisely we calculate the cohomology of the exceptional fiber E with coefficients in this exceptional sheaf, or—what comes to the same—the cohomology of $\tilde{X}$ with coefficients in the direct image sheaf $\mathscr{H}$ of $\tilde{\mathscr{F}} \mid E$. This calculation uses earlier results about local cohomology of form-rings of standard ideals [4].

In addition we determine the extremal *cohomological Hilbert functions* of the exceptional sheaf. This allows us in particular to express the degree of the exceptional sheaf in terms of the lengths of the local cohomology modules $H^i_p(\mathscr{F})$ of $\mathscr{F}$ at p.

In the next step we use the well known relation between $\tilde{\mathscr{F}}$ and the exceptional sheaf $\tilde{\mathscr{F}} \mid E$ to determine the cohomology of the blowing-up $\tilde{X}$ with coefficients in $\tilde{\mathscr{F}}$.

Then we apply our results in a special situation. We namely assume that X is regular, that $\mathscr{F}$ is locally free outside of p, and that the stalk $\mathscr{F}_p$ is a module of Buchsbaum type. We refer to this situation as the *B-singular* *case*. In this case, the blowing-up of $\tilde{X}$ at p has the standard-property. So we will consider this special blowing-up. In this case $\tilde{X}$ is regular and $\tilde{\tilde{\mathscr{F}}}$ is locally free. Moreover the exceptional fiber E is a projective space of dimension $d-1$ (over the function field $\kappa(p)$ of p), and the exceptional sheaf is a bundle over this projective space. We will now study this *exceptional bundle*, which turns out to be uniform. In particular we determine its splitting type.

Considering a blowing-up morphism $\pi: \tilde{X} \to X$ it seems fairly natural to look at the *pull-back* of the $\mathcal{O}_X$-sheaf $\mathscr{F}$, e.g., the inverse image sheaf $\pi^*\mathscr{F}$. The last section of this paper is therefore devoted to the comparison of the Rees modification $\tilde{\tilde{\mathscr{F}}}$ and the pull-back $\pi^*\mathscr{F}$. It turns out that $\tilde{\tilde{\mathscr{F}}}$ is obtained from $\pi^*\mathscr{F}$ by factoring out the torsion with respect to the exceptional divisor of π. In particular we get detailed information on the relation between $\tilde{\tilde{\mathscr{F}}}$ and $\pi^*\mathscr{F}$ in the case where X is smooth and $\mathscr{F}$ is B-singular. We namely may determine a lot of data of the kernel $\mathscr{K}$ of the canonical projection $\pi^*\mathscr{F} \mid E \to \tilde{\tilde{\mathscr{F}}} \mid E$.

For unexplained terminology and notations we refer to [16, 17, 19, 20].

2. Standard Blowing-Up

Let R be a noetherian ring, put $X = \operatorname{Spec}(R)$, and let $p \in X$ be a closed point. Let $\mathfrak{m} \subseteq R$ be the maximal ideal that corresponds to p. Let $\mathscr{F}$ be a coherent sheaf. $\mathscr{F}$ is induced by a finitely generated R-module M, and for the local cohomology modules we have the coincidence

$$H_p^i(\mathscr{F}) = H_{\mathfrak{m}}^i(M) \qquad (i = 0, 1, 2, ...).$$

Moreover $\dim_p(X) = \operatorname{ht}(\mathfrak{m}) =: d$.

We want to assume that $\mathscr{F}$ satisfies the *finiteness condition* at p, e.g.,

(2.1) $H_p^i(\mathscr{F}) = H_{\mathfrak{m}}^i(M)$ is finitely generated over R (or—equivalently—over $\mathcal{O}_{X,p}$) for all $i < d$.

A system $x_1, ..., x_d \in \mathfrak{m}$ is called a *standard-system* with respect to $\mathscr{F}$ at p, if $\mathfrak{m}$ is a minimal prime divisor of $(x_1, ..., x_d)$ and if

(2.2) $(x_1, ..., x_d)\, H_{\mathfrak{m}}^i(M/(x_{\sigma(1)}, ..., x_{\sigma(j)})M) = 0$ for all pairs $i, j \in \mathbb{N}_0$ with $i + j < d$ and all permutations σ of $\{1, ..., d\}$.

It is known, that the finiteness condition (2.1) implies the existence of standard-sequences (cf. [2, 24]).

(2.3) PROPOSITION. *There is a natural integer N such that each system $x_1, ..., x_d \in \mathfrak{m}^N$, for which $\mathfrak{m}$ is a minimal prime divisor of $(x_1, ..., x_d)$, is a standard-system with respect to $\mathscr{F}$ at p.*

(2.4) DEFINITION. An ideal $I \subseteq R$ is called a *standard-ideal* with respect to $\mathscr{F}$ at p, if there is a standard-sequence $x_1, ..., x_d$ with respect to $\mathscr{F}$ at p, such that

$$I_{\mathfrak{p}} = \begin{cases} (x_1, ..., x_d)_{\mathfrak{m}}, & \text{for } \mathfrak{p} = \mathfrak{m} \\ R_{\mathfrak{p}}, & \text{for } \mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}. \end{cases}$$

It is equivalent to say that I is the $\mathfrak{m}$-primary component of $(x_1, ..., x_d)$, where $x_1, ..., x_d$ is a standard-sequence.

For an ideal $I \subseteq R$ we consider the *Rees algebra*

$$\mathfrak{R}(I) = R \oplus I \oplus I^2 \oplus \cdots$$

and the *blowing-up*

$$\tilde{X} := \operatorname{Proj}(\mathfrak{R}(I)) \xrightarrow{\pi} X \tag{2.5}$$

of X at $\operatorname{Spec}(R/I)$.

(2.6) DEFINITION. If $I \subseteq R$ is a standard-ideal with respect to $\mathscr{F}$ at p, the blowing-up $\tilde{X} \xrightarrow{\pi} X$ is called a *standard blowing-up* with respect to $\mathscr{F}$ centered at p.

We now want to define the *Rees modification* $\tilde{\mathscr{F}}$ *of* $\mathscr{F}$. To do so, we introduce the *Rees module*

$$\mathfrak{R}(I, M) := \bigoplus_{n \geqslant 0} I^n M,$$

which we consider as a graded module over the Rees algebra $\mathfrak{R}(I)$. Then we put:

(2.7) DEFINITION. The *Rees modification* of $\mathscr{F}$ with respect to π is the coherent $\mathcal{O}_{\tilde{X}}$-sheaf induced by the Rees module of M:

$$\tilde{\mathscr{F}} := \mathfrak{R}(I, M)^{\sim}.$$

Clearly, putting $M = R$, we get $\tilde{\mathcal{O}}_X = \mathcal{O}_{\tilde{X}}$.

Next we introduce the *exceptional fiber*

$$E := \operatorname{Proj}(\operatorname{Gr}(I)) \subseteq \tilde{X}$$

of the blowing-up $\pi: \tilde{X} \to X$, which is defined by the *associated graded ring*

$$\mathrm{Gr}(I) = R/I \otimes_R \mathfrak{R}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

(Remember that E coincides with $\pi^{-1}(\mathrm{Spec}(R/I))$ as a set, and that the restriction $\pi| : \tilde{X} - E \to X - \mathrm{Spec}(R/I)$ is an isomorphism.) The study of $\tilde{\mathscr{F}}$ heavily relies on the study of the restriction $\tilde{\mathscr{F}} \mid E$ of that sheaf to the exceptional fiber E. We call $\tilde{\mathscr{F}} \mid E$ the *exceptional sheaf*. So we will give another description of this latter sheaf.

To do so we introduce the *associated graded module*

$$\mathrm{Gr}(I, M) := R/I \otimes_R \mathfrak{R}(I, M) = \bigoplus_{n \geq 0} I^n M/I^{n+1} M.$$

If we consider $\mathrm{Gr}(I, M)$ as a graded module over $\mathrm{Gr}(I)$, it induces a coherent sheaf over E. This induced sheaf is nothing else than the exceptional sheaf

(2.8) $$\tilde{\mathscr{F}} \mid E = \mathrm{Gr}(I, M)^{\sim}.$$

We also may consider $\mathrm{Gr}(I, M)$ as a graded module over the Rees algebra $\mathfrak{R}(I)$. This module then induces a coherent sheaf over $\tilde{X}$, which we denote by $\mathscr{H}$,

(2.9) $$\mathscr{H} := \mathrm{Gr}(I, M)^{\sim}, \qquad (\text{over } X^{\sim} = \mathrm{Proj}(\mathfrak{R}(I))).$$

Denoting the inclusion map $E \to \tilde{X}$ by j, we may write $\mathscr{H}$ as the direct image of the exceptional sheaf under j

(2.10) $$\mathscr{H} = j_*(\tilde{\mathscr{F}} \mid E).$$

Our proceeding to calculate the cohomology modules (1.2) is based on a preliminary calculation of the cohomology of E with coefficients in $\mathscr{H}$. Then, to go on, we need a relation between the two sheaves $\tilde{\mathscr{F}}$ and $\mathscr{H}$. We next shall work out this relation.

On the category of coherent sheaves over $\tilde{X}$ we have the operation of *twisting*, given by taking tensor products with the very ample sheaf (cf. [19, 26])

$$\mathscr{O}_{\tilde{X}}(1) := (\mathfrak{R}(I)(1))^{\sim},$$

in which $\mathfrak{R}(I)(1)$ is obtained by shifting $\mathfrak{R}(I)$ one place to the left.

Now, we may formulate the announced relation:

(2.11) Proposition. *For each $n \in \mathbb{Z}$ there is a short exact sequence*

$$0 \to \tilde{\mathscr{F}}(n+1) \to \tilde{\mathscr{F}}(n) \to \mathscr{H}(n) \to 0.$$

Proof. As twisting is an exact functor, we may restrict ourselves to the case $n=0$. We have the following short exact sequences of graded $\mathfrak{R}(I)$-modules

$$0 \longrightarrow I\mathfrak{R}(I, M) \xrightarrow{\text{incl.}} \mathfrak{R}(I, M) \longrightarrow \mathrm{Gr}(I, M) \longrightarrow 0$$

$$0 \longrightarrow I\mathfrak{R}(I, M)(-1) \xrightarrow{\text{incl.}} \mathfrak{R}(I, M) \longrightarrow M \longrightarrow 0,$$

where—in the second sequence—M is considered as a graded $\mathfrak{R}(I)$-module concentrated to degree 0. Passing to the induced coherent sheaves of $\mathcal{O}_{\tilde{X}}$-modules, we obtain exact sequences

$$0 \to I\mathfrak{R}(I, M)^{\sim} \to \tilde{\mathscr{F}} \to \mathscr{H} \to 0$$

$$0 \to I\mathfrak{R}(I, M)^{\sim}\,(-1) \to \tilde{\mathscr{F}} \to M^{\sim} \to 0.$$

As M is concentrated to one degree, it is annihilated by the irrelevant ideal $\mathfrak{R}(I)_{>0}$ of $\mathfrak{R}(I)$. Therefore we have $M^{\sim}=0$, and so the second sequence yields an isomorphism $I\mathfrak{R}(I, M)^{\sim}\,(-1) \cong \tilde{\mathscr{F}}$. Twisting by 1 furnishes an isomorphism $I\mathfrak{R}(I, M)^{\sim} \cong \tilde{\mathscr{F}}(1)$. Altogether we obtain an exact sequence

$$0 \to \tilde{\mathscr{F}}(1) \to \tilde{\mathscr{F}} \to \mathscr{H} \to 0,$$

hence our claim. ∎

3. Cohomology of the Exceptional Fiber

Throughout this section, let $X=\mathrm{Spec}(R)$ be a noetherian scheme, let $p \in X$ be a closed point corresponding to the maximal ideal $\mathfrak{m} \subseteq R$, and let $\mathscr{F}$ be a coherent sheaf over X induced by a finitely generated R-module M. Moreover let $\dim_p(X)=\mathrm{ht}(\mathfrak{m})=:d>1$ and assume that $\mathscr{F}$ satisfies the finiteness condition (2.1) at p.

Finally let $I \subseteq R$ be a standard-ideal with respect to $\mathscr{F}$ at p and consider the corresponding standard blowing-up

$$\tilde{X} := \mathrm{Proj}(\mathfrak{R}(I)) \xrightarrow{\pi} X.$$

In this situation the exceptional fiber of π is given as a set by $E=\pi^{-1}(p)$ and the restriction $\pi| : \tilde{X}-E \to X-\{p\}$ is an isomorphism.

Now, define the sheaf $\mathscr{H}$ according to (2.9) (or according to (2.10)). The goal of this section is to calculate the cohomology of $\tilde{X}$ with coefficients in $\mathscr{H}$ or—what comes to the same—the cohomology of the exceptional fiber E with coefficients in the exceptional sheaf $\tilde{\mathscr{F}} \mid E$.

We begin with a general remark. So let $\mathscr{G}$ be an arbitrary coherent sheaf over $\tilde{X}$. Then the direct sum

$$H^i_*(\tilde{X}, \mathscr{G}) := \bigoplus_{n \in \mathbb{Z}} H^i(\tilde{X}, \mathscr{G}(n)) \qquad (i = 0, 1, 2, ...)$$

naturally carries the structure of a graded $\mathfrak{R}(I)$-module.

$H^i_*(\tilde{X}, \mathscr{G})$ is called the ith *total cohomology module of* $\tilde{X}$ *with coefficients in* $\mathscr{G}$.

If $x \in I^n$ we write $x^{(n)}$ for the element x considered as a form of degree n in $\mathfrak{R}(I)$. Using this notation we may say that the $\mathfrak{R}(I)$-module $H^i_*(\tilde{X}, \mathscr{G})$ is known, if we know all the R-modules $H^i(\tilde{X}, \mathscr{G}(n))$ and all the R-linear maps

$$x^{(1)}: H^i(\tilde{X}, \mathscr{G}(n)) \to H^i(\tilde{X}, \mathscr{G}(n+1)) \qquad (x \in I, n \in \mathbb{Z}).$$

We now give another description of the modules $H^i_*(\tilde{X}, \mathscr{G})$. To do so, we assume that $\mathscr{G}$ is induced by a finitely generated, graded $\mathfrak{R}(I)$-module $N = \bigoplus_{n \in \mathbb{Z}} N_n$:

$$\mathscr{G} = N^{\sim}.$$

Moreover, for an arbitrary noetherian ring A and an ideal $J \subseteq A$ write H^j_J for the jth *local cohomology* functor with respect to J and D_J for the functor of *J-transformation*. This latter is defined as

$$D_J(\cdot) = \varinjlim_n \operatorname{Hom}_A(J^n, \cdot).$$

Using these notations and writing $\mathfrak{R}(I)_{>0}$ for the irrelevant ideal of $\mathfrak{R}(I)$ we have graded natural isomorphisms (cf. [18])

$$(3.1) \qquad \begin{aligned} &\text{(i)} \quad H^0_*(\tilde{X}, \mathscr{G}) \cong D_{\mathfrak{R}(I)_{>0}}(N). \\ &\text{(ii)} \quad H^i_*(\tilde{X}, \mathscr{G}) \cong H^{i+1}_{\mathfrak{R}(I)_{>0}}(N) \qquad (i > 0). \end{aligned}$$

Finally we introduce the following notation: If T is an R-module and if $r \in \mathbb{Z}$, $T^{(r)}$ stands for the graded $\mathfrak{R}(I)$-module which consists in T concentrated to degree r. So the nth homogeneous part $T^{(r)}_n$ of $T^{(r)}$ is given by

$$T^{(r)}_n = \begin{cases} 0, & \text{for} \quad n \neq r \\ T, & \text{for} \quad n \neq r. \end{cases}$$

Using this notation we have the following auxiliary result, which will play a crucial role in this section:

(3.2) LEMMA. $H^i_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) = H^i_p(\mathcal{F})^{(-i)}$ *for* $i \leq d-1$.

Proof. By [4] we know that

$$H^i_{(\mathfrak{m}, \mathfrak{R}(I)_{>0})}(\mathrm{Gr}(I, M)) = H^i_{\mathfrak{m}}(M)^{(i)} \qquad \text{for} \quad i = 0, ..., d-1.$$

As $\mathfrak{m}$ is the unique prime divisor of I we have $\mathfrak{m} = \sqrt{I}$. So there is an $r \in \mathbb{N}$ with $\mathfrak{m}^r \subseteq I$. As a consequence we have $\mathfrak{m}^r \cdot (I^n M / I^{n+1} M) = 0$, $\forall n \in \mathbb{N}_0$, thus $\mathfrak{m}^r \, \mathrm{Gr}(I, M) = 0$. Therefore we may write

$$H^i_{(\mathfrak{m}, \mathfrak{R}(I)_{>0})}(\mathrm{Gr}(I, M)) = H^i_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M).$$

Observing $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}}(M) = H^i_p(\mathcal{F})$, we get our claim. ▮

As a first consequence of (3.2) we obtain

(3.3) PROPOSITION. $H^i_*(\tilde{X}, \mathcal{H}) = H^{i+1}_p(\mathcal{F})^{(-i-1)}$ *for* $i = 1, ..., d-2$.

Proof. Apply (3.1)(ii) to $\mathcal{H} = \mathrm{Gr}(I, M)^{\sim}$ and use (3.2). ▮

Further we have

(3.4) COMPLEMENT. $H^i_*(\tilde{X}, \mathcal{H}) = 0$ *for* $i \geq d$.

Proof. Use $H^i_*(\tilde{X}, \mathcal{H}) = H^i_*(E, \tilde{\mathcal{F}} \mid E)$ and observe that $\dim(E) = d-1$. ▮

In view of (3.3) and (3.4) it remains to determine the modules $H^0_*(\tilde{X}, \mathcal{H})$ and $H^{d-1}_*(\tilde{X}, \mathcal{H})$. We begin with the study of the *total module of global sections* $H^0_*(\tilde{X}, \mathcal{H})$.

To do so, we first put

$$\bar{M} := M / H^0_{\mathfrak{m}}(M).$$

Then $H^1_{\mathfrak{m}}(\bar{M}) = H^1_{\mathfrak{m}}(M) = H^1_p(\mathcal{F})$. By the standard property of I we have $IH^1_p(\mathcal{F}) = 0$. So choosing $x \in I$ and making use of the short exact sequence

$$0 \longrightarrow \bar{M} \longrightarrow D_{\mathfrak{m}}(\bar{M}) \xrightarrow{\beta} H^1_{\mathfrak{m}}(\bar{M}) \longrightarrow 0$$

we obtain $xD_{\mathfrak{m}}(\bar{M}) \subseteq \bar{M}$. This allows us to define a map

$$x\colon H^1_p(\mathcal{F}) \to \bar{M}/I\bar{M} \tag{3.5}$$

by $a \mapsto x\beta^{-1}(a)/I\bar{M}$ $(a \in H^1_p(\mathcal{F}) = H^1_{\mathfrak{m}}(\bar{M}))$.

This map will be used to describe the action of the 1-forms $x^{(1)} \in \mathfrak{R}(I)$ on $H^0_*(\tilde{X}, \mathcal{H})$.

Next, we make another remark. By the standard-property of I we have (cf. [2])

$$H^0_{\mathfrak{m}}(M) \cap IM = 0.$$

As a consequence we obtain canonical isomorphisms

$$(3.6)\qquad I^nM/I^{n+1}M \xrightarrow{\cong} I^n\bar{M}/I^{n+1}\bar{M} \qquad (n>0).$$

Now, we may describe $H^0_*(\tilde{X}, \mathscr{H})$.

(3.7) PROPOSITION.

(i) $$H^0(\tilde{X}, \mathscr{H}(n)) = \begin{cases} 0, & \textit{for}\quad n<-1 \\ H^1_p(\mathscr{F}), & \textit{for}\quad n=-1 \\ \bar{M}/I\bar{M}, & \textit{for}\quad n=0 \\ I^nM/I^{n+1}M, & \textit{for}\quad n>0. \end{cases}$$

(ii) *For $x \in I$, the multiplication map $x^{(1)}$: $H^0(\tilde{X}, \mathscr{H}(-1)) \to H^0(\tilde{X}, \mathscr{H})$ is exactly the map x: $H^1_p(\mathscr{F}) \to \bar{M}/I\bar{M}$ defined in* (3.5).

(iii) *For $x \in I$ and for $n \geqslant 0$, the multiplication maps $x^{(1)}$: $H^0(\tilde{X}, \mathscr{H}(n)) \to H^0(\tilde{X}, \mathscr{H}(n+1))$ are the canonical multiplication maps x: $I^n\bar{M}/I^{n+1}\bar{M} \to I^{n+1}\bar{M}/I^{n+2}\bar{M}$.*

Proof. There is a canonical graded exact sequence [3]

$$0 \to H^0_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) \to \mathrm{Gr}(I, M) \to D_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) \\ \to H^1_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) \to 0.$$

In the proof of (3.2) we have seen that $H^0_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) = H^0_{\mathfrak{m}}(M)^{(0)}$ and $H^1_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) = H^1_p(\mathscr{F})^{(-1)}$.

Applying (3.1)(i) with $N = \mathrm{Gr}(I, M)$ we thus get a canonical graded sequence

$$0 \to H^0_{\mathfrak{m}}(M)^{(0)} \to \mathrm{Gr}(I, M) \to H^0_*(\tilde{X}, \mathscr{H}) \to H^1_p(\mathscr{F})^{(-1)} \to 0.$$

Moreover we have $\mathrm{Gr}(I, M)_0/H^0_{\mathfrak{m}}(M) = M/(IM + H^0_{\mathfrak{m}}(M)) = \bar{M}/I\bar{M}$. In view of (3.6) we have $\mathrm{Gr}(I, M)_n = I^n\bar{M}/I^{n+1}\bar{M}$ for all $n>0$. Altogether, this proves our claim. ∎

Now, we have calculated the modules

$$H^i_*(X, \mathscr{H}) = H^i_*(E, \bar{\mathscr{F}} \mid E) \qquad \text{for}\quad i \neq d.$$

In particular we know:

(3.8) COROLLARY.

(i) $$H^0(\tilde{X}, \mathscr{H}(n)) = \begin{cases} 0, & \textit{for}\quad n<-1 \\ H^1_p(\mathscr{F}), & \textit{for}\quad n=-1 \\ \bar{M}/I\bar{M}, & \textit{for}\quad n=0 \\ I^nM/I^{n+1}M, & \textit{for}\quad n>0. \end{cases}$$

(ii) $H^i(\tilde{X}, \mathscr{H}(n)) = \begin{cases} H_p^{i+1}(\mathscr{F}), & \text{for } n = -i-1 \\ 0, & \text{for } n \neq -i-1, \end{cases}$

for $1 \leqslant i < d-1$.

(iii) $H^i(\tilde{X}, \mathscr{H}(n)) = 0$, *for all* $n \in \mathbb{Z}$ *and all* $i \geqslant d$.

So, it remains to treat the module $H_*^{d-1}(\tilde{X}, \mathscr{H})$. Unfortunately we are not able to describe this module in an explicit way as above. But at least we may calculate the lengths of the single groups $H^{d-1}(\tilde{X}, \mathscr{H}(n))$. This will be the task of the next section.

4. The Lengths of the Modules $H^{d-1}(\tilde{X}, \mathscr{H}(n))$

We keep the notations and hypotheses of the previous section. If T is an R-module, we denote the (possibly infinite) *length* of T by $l(T)$. Using this notation, we define

$$(4.1) \qquad h^i(\tilde{X}, \mathscr{G}) := l(H^i(\tilde{X}, \mathscr{G})); \qquad h_p^i(\mathscr{F}) := l(H_p^i(\mathscr{F})),$$

where $\mathscr{G}$ is a coherent sheaf over $\tilde{X}$. By our finiteness condition (2.1) (and by the fact that $H_p^i(\mathscr{F})$ is artinian) we know that $h_p^i(\mathscr{F})$ is finite for all $i \neq d$.

The goal of this section is to express the $(d-1)$st *cohomological Hilbert function* defined by $n \mapsto h^{d-1}(\tilde{X}, \mathscr{H}(n))$ in terms of the numbers $h_i^p(\mathscr{F})$ for $i < d$.

Our investigation only concerns the module $\mathrm{Gr}(I, M)$. The homogeneous parts $I^nM/I^{n+1}M$ of this module do not change, if we localize at $\mathfrak{m}$, as they are supported by p. This allows us to replace R by $R_{\mathfrak{m}}$, thus to assume that $(R, \mathfrak{m})$ is local. We will keep this assumption throughout this section.

We will describe the functions $h^{d-1}(\tilde{X}, \mathscr{H}(n))$ by means of a recursive procedure. To get this description we need some preliminary reduction results, which allow us to replace d by $d-1$.

We may write $I = (x_1, ..., x_d)$, where $x_1, ..., x_d$ is a fixed standard-system with respect to $\mathscr{F}$ at p. We put

$$(4.2) \qquad \begin{aligned} &\text{(i)} \quad R' := R/x_1R;\ x_i' := x_i/x_1R \in R' \quad (i = 2, ..., d) \\ &\text{(ii)} \quad I' := IR' = (x_2', ..., x_d') \\ &\text{(iii)} \quad M' := M/x_1M \\ &\text{(iv)} \quad X' := \mathrm{Spec}(R'),\ \mathscr{F}' := \mathscr{F} \mid X' = \tilde{M}'. \end{aligned}$$

Then clearly X' is a closed subscheme of X, which satisfies $\dim_p(X') = d-1$. Moreover $x'_2, ..., x'_d$ is a standard-system with respect to $\mathscr{F}'$ at p. Therefore the morphism

$$\pi' : \tilde{X}' = \operatorname{Proj}(\mathfrak{R}(I')) \to X' \tag{4.3}$$

is a standard blowing-up with respect to $\mathscr{F}'$ centered at p. Finally we introduce the coherent $\mathcal{O}_{\tilde{X}'}$ sheaf

$$\mathscr{H}' = \operatorname{Gr}(I', M')^{\sim}. \tag{4.4}$$

The modules $H^i(\tilde{X}', \mathscr{H}'(n))$ may be considered as such over R or over R'.

(4.5) LEMMA. *For each $n \in \mathbb{Z}$ there is an exact sequence of R-modules*

$$\begin{aligned} 0 &\longrightarrow H^0(\tilde{X}, \mathscr{H}(n)) \xrightarrow{x_1^{(1)}} H^0(\tilde{X}, \mathscr{H}(n+1)) \longrightarrow H^0(\tilde{X}', \mathscr{H}'(n+1)) \\ &\longrightarrow H^1(\tilde{X}, \mathscr{H}(n)) \xrightarrow{x_1^{(1)}} H^1(\tilde{X}, \mathscr{H}(n+1)) \longrightarrow H^1(\tilde{X}', \mathscr{H}'(n+1)) \\ &\cdots H^i(\tilde{X}, \mathscr{H}(n)) \xrightarrow{x_1^{(1)}} H^i(\tilde{X}, \mathscr{H}(n+1)) \longrightarrow H^i(\tilde{X}', \mathscr{H}'(n+1)) \\ &\longrightarrow H^{i+1}(\tilde{X}, \mathscr{H}(n)) \cdots . \end{aligned}$$

Proof. By the exactness of twisting we may assume that $n = 0$. Moreover—as $\operatorname{Gr}(I, M)$ and $\operatorname{Gr}(I, \bar{M})$ differ only in degree 0 (cf. (3.6))—we have $\mathscr{H} = \operatorname{Gr}(I, \bar{M})^{\sim}$. This allows us to replace M by $\bar{M}$, thus to assume that $H^0_{\mathfrak{m}}(M) = 0$. But then, there is an exact sequence

$$0 \longrightarrow \operatorname{Gr}(I, M) \xrightarrow{x_1^{(1)}} \operatorname{Gr}(I, M)(1) \longrightarrow \operatorname{Gr}(I', M')(1) \longrightarrow 0$$

of graded modules [4].

Passing to the induced $\mathcal{O}_{\tilde{X}}$-sheaves, we get an exact sequence

$$0 \longrightarrow \mathscr{H} \xrightarrow{x_1^{(1)}} \mathscr{H}(1) \longrightarrow j_* \mathscr{H}'(1) \longrightarrow 0,$$

where $j: \tilde{X}' \to \tilde{X}$ is the inclusion map. As j is a closed immersion, we have $H^i(\tilde{X}, j_* \mathscr{H}'(1)) = H^i(\tilde{X}', \mathscr{H}'(1))$. So, passing to cohomology, we get the requested sequence. ∎

This lemma will be an essential tool to make induction. To have started induction, we need the following result.

(4.6) LEMMA. *For $d = 1$ we have $H^0(\tilde{X}, \mathscr{H}(n)) = \bar{M}/I\bar{M}$ for all $n \in \mathbb{Z}$. Morever the multiplication maps*

$$x_1^{(1)} : H^0(\tilde{X}, \mathscr{H}(n)) \to H^0(\tilde{X}, \mathscr{H}(n))$$

are isomorphisms.

For $i > 0$, $H^i(\tilde{X}, \mathscr{H}(n))$ vanishes for all $n \in \mathbb{Z}$.

Proof. We may write $I=(x_1)$. Thereby x_1 is a non-zero divisor with respect to M. Therefore we have isomorphisms $\operatorname{Gr}(I, \bar{M})_n = I^n\bar{M}/I^{n+1}\bar{M} = x_1^n\bar{M}/x_1^{n+1}\bar{M} \cong \bar{M}/x_1\bar{M} = \bar{M}/I\bar{M}$ for all $n \geqslant 0$. Moreover $x_1^{(1)}\colon \operatorname{Gr}(I, \bar{M})_n \to \operatorname{Gr}(I, \bar{M})_{n+1}$ is an isomorphism for all $n \geqslant 0$. As $\mathscr{H}$ is induced by $\operatorname{Gr}(I, \bar{M})$, the graded $\mathfrak{R}(I)$-modules $H^0_*(\tilde{X}, \mathscr{H})$ and $\operatorname{Gr}(I, \bar{M})$ coincide in large degrees. As $\mathscr{H}$ is supported by finitely many points only, it is not affected by twisting and has vanishing higher cohomology. This proves our claim.

We shall also use the following result, in which $\mathscr{F}'$ is defined according to (4.2)

(4.7) LEMMA. *Let $i \in \{1, ..., d-1\}$. Then*

$$h_p^{i-1}(\mathscr{F}') = h_p^i(\mathscr{F}) + h_p^{i-1}(\mathscr{F}).$$

Proof. As x_1 is a non-zero divisor with respect to $\bar{M}$ [2], we have the following diagram in which the first row is exact:

$$\begin{tikzcd} 0 \arrow[r] & \bar{M} \arrow[r] \arrow[dr, phantom, "\circlearrowleft\ \mathrm{can.}"] & M \arrow[r] & M/x_1M \arrow[r] & 0. \\ & M \arrow[u, "\gamma"] \arrow[ur, "x_1"'] & & & \end{tikzcd}$$

Passing to the local cohomology functors $H^i_{\mathfrak{m}}$, observing that $H^0_{\mathfrak{m}}(\bar{M})=0$, that $H^i_{\mathfrak{m}}(\gamma)$ is an isomorphism for $i>0$, and that $x_1H^i_{\mathfrak{m}}(M)=0$ for all $i<d$, we get exact sequences

$$0 \to H^{i-1}_{\mathfrak{m}}(M) \to H^{i-1}_{\mathfrak{m}}(M/XM) \to H^i_{\mathfrak{m}}(M) \to 0 \qquad (i<d).$$

Now, we conclude by $H^j_{\mathfrak{m}}(M) = H^j_p(\mathscr{F})$, $H^{i-1}_{\mathfrak{m}}(M/xM) = H^{i-1}_p(\mathscr{F}')$. ∎

As a last auxiliary result we prove

(4.8) LEMMA. *Let $d>1$. Then $H^{d-1}(\tilde{X}, \mathscr{H}(n))=0$ for all $n>-d$.*

Proof. In view of (3.1)(ii) we have to show that $H^d_{\mathfrak{R}(I)_{>0}}(\operatorname{Gr}(I, M))$ vanishes in all degrees $>-d$. We do this by induction, starting with $d=1$. Again we may assume that $H^0_{\mathfrak{m}}(M)=0$, just by replacing M by $\bar{M}$. Then $H^0_{\mathfrak{R}(I)_{>0}}(\operatorname{Gr}(I, M)) = H^0_{\mathfrak{m}}(M)^{(0)}=0$ gives rise to a graded exact sequence [3]

$$0 \longrightarrow \operatorname{Gr}(I, M) \stackrel{\iota}{\longrightarrow} D_{\mathfrak{R}(I)_{>0}}(\operatorname{Gr}(I, M))$$
$$\stackrel{\Psi}{\longrightarrow} H^1_{\mathfrak{R}(I)_{>0}}(\operatorname{Gr}(I, M)) \longrightarrow 0.$$

By (3.1)(ii) the middle term may be replaced by $H^0_*(\tilde{X}, \mathscr{H})$. So, by (4.6), ι is an isomorphism in all degrees $\geqslant 0$. Therefore $H^0_{\mathfrak{R}(I)_{>0}}(\operatorname{Gr}(I, M))$ vanishes in all degrees >-1.

Now, let $d>1$. Again we may suppose $H^0_{\mathfrak{m}}(M)=0$. Then, the graded exact sequence

$$0 \longrightarrow \mathrm{Gr}(I, M) \xrightarrow{x_1^{(1)}} \mathrm{Gr}(I, M)(1) \longrightarrow \mathrm{Gr}(I', M')(1) \longrightarrow 0$$

(cf. proof of (4.5)) induces an exact sequence

$$H^{d-1}_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I', M'))(1) \longrightarrow H^d_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) \xrightarrow{x_1^{(1)}} H^d_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M))(1).$$

By induction, the left-hand module vanishes in degrees $>-d$. So, the multiplication map $x_1^{(1)}$ is injective in degrees $>d$. As $H^d_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M))$ is an $x_x^{(1)}$-torsion-module, it therefore must vanish in degrees $>-d$. ∎

Now, to formulate our main result, we define functions

$$s_j^{(d)}\colon \mathbb{Z} \to \mathbb{N}_0 \qquad (d \in \mathbb{N};\, j=0, ..., d-1)$$

by the following recursive procedure:

(4.9) (i) $s_0^{(1)}(n)=1,\ \forall n \in \mathbb{Z}$.

(ii) If $d>1$, then

(a) $$s_0^{(d)}(n) := \begin{cases} 0, & \text{for } n>-d \\ \sum\limits_{n<k\leqslant -d+1} s_0^{(d-1)}(k), & \text{for } n\leqslant -d, \end{cases}$$

(b) $$s_j^{(d)}(n) := \begin{cases} 0, & \text{for } n>-d \\ \sum\limits_{n<k\leqslant -d+1} (s_{j-1}^{(d-1)}+s_j^{(d-1)})(k), & \text{for } n\leqslant -d \\ (j=1, ..., d-2), & \end{cases}$$

(c) $$s_{d-1}^{(d)}(n) := \begin{cases} 0, & \text{for } n>-d \\ 1+\sum\limits_{n<k\leqslant -d+1} s_{d-2}^{(d-1)}(k), & \text{for } n\leqslant -d. \end{cases}$$

Using the Pascal formulas for binomial coefficients we get

(4.10) $$s_0^{(d)}(n)=\binom{-n-1}{d-1} \qquad \text{for } n\leqslant -d.$$

Now, we are ready to state:

(4.11) Proposition. *For $d \geqslant 1$, the length of the cohomology module $H^{d-1}(\tilde{X}, \mathscr{H}(n))$ is given by*

$$h^{d-1}(\tilde{X}, \mathscr{H}(n)) = s_0^{(d)}(n)\, l(M/IM) - \sum_{j=0}^{d-1} s_j^{(d)}(n)\, h_p^j(\mathscr{F}).$$

Proof (*Induction on d*). For $d=1$ we have $h^0(\tilde{X}, \mathscr{H}(n)) = l(\bar{M}/I\bar{M})$ (cf. (4.6)). So we may write $h^0(\tilde{X}, \mathscr{H}(n)) = l(\bar{M}/I\bar{M}) = l(M/(IM + H_p^0(\mathscr{F})))$ $= l(M/IM) - l(IM + H_p^0(\mathscr{F})/IM)$. As $H_p^0(\mathscr{F}) \cap IM = 0$, we have $l(IM + H_p^0(\mathscr{F})/IM) = l(H_p^0(\mathscr{F})) = h_p^0(\mathscr{F})$, thus $h^0(\tilde{X}, \mathscr{H}(n)) = l(M/IM) - h_p^0(\mathscr{F})$. This proves the case $d=1$.

So, let $d > 1$. By (3.8)(ii), (iii), and by (4.5) there are exact sequences

$$0 \to H^{d-2}(\tilde{X}, \mathscr{H}(n+1)) \to H^{d-2}(\tilde{X}', \mathscr{H}'(n+1))$$
$$\to H^{d-1}(\tilde{X}, \mathscr{H}(n)) \to H^{d-1}(\tilde{X}, \mathscr{H}(n+1)) \to 0.$$

For $n > -d$, $H^{d-1}(\tilde{X}, \mathscr{H}(n))$ vanishes by (4.8). So let $n \leqslant -d$. By induction we have $h^{d-2}(\tilde{X}', \mathscr{H}'(k)) = s_0^{(d-1)}(k)\, l(M'/IM') - \sum_{j=0}^{d-2} s_j^{(d-1)}(k)\, h_p^j(\mathscr{F}')$. Observing that $M'/IM' \cong M/IM$ and making use of (4.7) we thus may write

$$\begin{aligned} h^{d-2}(\tilde{X}', \mathscr{H}'(k)) = {} & s_0^{d-1}(k)\, l(M/IM) - s_0^{(d-1)}(k)\, h_p^0(\mathscr{F}) \\ & - \sum_{j=1}^{d-2} (s_{j-1}^{(d-1)} + s_j^{(d-1)})(k)\, h_p^j(\mathscr{F}) \\ & - s_{d-2}^{(d-1)}(k)\, h_p^{d-1}(\mathscr{F}). \end{aligned}$$

Choosing $n = -d$ and observing that $H^{d-2}(\tilde{X}, \mathscr{H}(-d+1)) = h_p^{d-1}(\mathscr{F})$ (cf. (3.8)) we get from the previous sequence

$$\begin{aligned} & h^{d-1}(\tilde{X}, \mathscr{H}(-d)) \\ & = h^{d-2}(\tilde{X}', \mathscr{H}'(-d+1)) - h_d^{d-1}(\mathscr{F}) \\ & = s_0^{(d-1)}(-d+1)\, l(M/IM) - s_0^{(d-1)}(-d+1)\, h_p^0(\mathscr{F}) \\ & \quad - \sum_{j=1}^{d-2} (s_{j-1}^{(d-1)} + s_j^{(d-1)})(-d+1)\, h_p^j(\mathscr{F}) \\ & \quad - (s_{d-2}^{(d-1)}(-d+1) + 1)\, h_p^{d-1}(\mathscr{F}). \end{aligned}$$

Using the same sequence, and observing that $H^{d-2}(\tilde{X}, \mathscr{H}(n+1))$ vanishes for $n < -d$ (cf. (3.8)) we get $h^{d-1}(\tilde{X}, \mathscr{H}(n)) = h^{d-1}(\tilde{X}, \mathscr{H}(-d)) + \sum_{n < k \leqslant -d+1} h^{d-2}(\tilde{X}', \mathscr{H}'(k))$ for $n < -d$. Expressing $h^{d-1}(\tilde{X}, \mathscr{H}(-d))$ and $h^{d-2}(\tilde{X}', \mathscr{H}'(k))$ by the previous formulas and using (4.9), we get our claim. ∎

5. The Hilbert Function $h^0(\tilde{X}, \mathscr{H}(n))$

In the previous section we have calculated the $(d-1)$st cohomological Hilbert function $h^{d-1}(\tilde{X}, \mathscr{H}(n))$. To know all the cohomological Hilbert functions $h^i(\tilde{X}, \mathscr{H}(n))$ of $\mathscr{H}$ we have to determine in addition the 0th of these functions, namely $h^0(\tilde{X}, \mathscr{H}(n))$. It should be noted that the cases $i \neq 0, d-1$ are immediate from (3.8).

We keep the notations and hypotheses of the previous section. By (3.8)(i) we know that for $d > 1$

$$h^0(\tilde{X}, \mathscr{H}(n)) = \begin{cases} 0, & \text{if } n < -1 \\ h_p^1(\mathscr{F}), & \text{if } n = -1. \end{cases} \tag{5.1}$$

Moreover we have (again for $d > 1$)

$$h^0(\tilde{X}, \mathscr{H}(0)) = l(M/IM) - h_p^0(\mathscr{F}). \tag{5.2}$$

Indeed, by (3.8) we have $h^0(\tilde{X}, \mathscr{H}(0)) = l(\bar{M}/I\bar{M})$. But this latter length has been shown to equal $l(M/IM) - h_p^0(\mathscr{F})$ in the proof of (4.11).

So it remains to calculate the values of $h^0(\tilde{X}, \mathscr{H}(n))$ for $n > 0$. To do so, we define functions

$$t_j^{(d)}: \mathbb{Z} \to \mathbb{N}_0 \qquad (d \in \mathbb{N};\, j = 0, ..., d-1)$$

in a similar way as we did in (4.9):

(5.3)

(i) $t_0^{(1)}(n) = 1, \forall n \in \mathbb{Z}$.

(ii) If $d > 1$, then:

$$\text{(a)} \quad t_0^{(d)}(n) := \begin{cases} 0, & \text{for } n \leqslant 0 \\ 1 + \sum\limits_{0 < k \leqslant n} t_0^{(d-1)}(k), & \text{for } n > 0 \end{cases}$$

$$\text{(b)} \quad t_j^{(d)}(n) := \begin{cases} 0, & \text{for } n \leqslant 0 \\ \sum\limits_{0 < k \leqslant n} (t_{j-1}^{(d-1)} + t_j^{(d-1)})(k), & \text{for } n > 0 \\ (j = 1, ..., d-2) & \end{cases}$$

$$\text{(c)} \quad t_{d-1}^{(d)}(n) := \begin{cases} 0, & \text{for } n \leqslant 0 \\ \sum\limits_{0 < k \leqslant n} t_{d-2}^{(d-1)}(k), & \text{for } n > 0. \end{cases}$$

Again, $t_0^{(d)}$ may be expressed in terms of binomial coefficients

$$t_0^{(d)}(n) = \binom{n+d-1}{d-1} \qquad \text{for } n > 0. \tag{5.4}$$

Using these notations we have

(5.5) PROPOSITION. *For $d>1$, the length of the module of global sections $H^0(\tilde{X}, \mathscr{H}(n))$ is given by*

$$h^0(\tilde{X}, \mathscr{H}(n)) = \begin{cases} 0, & \text{for } n<-1 \\ h_p^1(\mathscr{F}), & \text{for } n=-1 \\ l(M/IM) - h_p^0(\mathscr{F}), & \text{for } n=0 \\ t_0^{(d)}(n)\, l(M/IM) - \sum_{j=0}^{d-1} t_j^{(d)}(n)\, h_p^j(\mathscr{F}), & \text{for } n>0. \end{cases}$$

Proof (*Induction on d*). Let $d=2$. Defining $\tilde{X}'$, $\mathscr{H}'$, M', and $\mathscr{F}'$ as in Section 4, we have $h^0(\tilde{X}', \mathscr{H}'(n)) = l(M'/I'M') - h_p^0(\mathscr{F}')$ for all $n \in \mathbb{Z}$ (cf. (4.11)). Applying (4.7) and observing that $l(M'/I'M') = l(M/IM)$, we thus may write $h^0(\tilde{X}', \mathscr{H}'(n)) = l(M/IM) - h_p^0(\mathscr{F}) - h_p^1(\mathscr{F})$.

Applying (4.5) with $n=k-1 \geqslant 0$ we get

$$\begin{aligned} h^0(\tilde{X}, \mathscr{H}(k)) &= h^0(\tilde{X}, \mathscr{H}(k-1)) + l(M/IM) - h_p^0(\mathscr{F}) - h_p^1(\mathscr{F}) \\ &= h^0(\tilde{X}, \mathscr{H}(k-1)) + t_0^{(1)}(k)\, l(M/IM) - t_0^{(1)}(k)\, h_p^0(\mathscr{F}) \\ &\quad - t_0^{(1)}(k)\, h_p^1(\mathscr{F}). \end{aligned}$$

As $h^0(\tilde{X}, \mathscr{H}(0)) = l(M/IM) - h_p^0(\mathscr{F})$ we thus obtain for $n>0$

$$\begin{aligned} h^0(\tilde{X}, \mathscr{H}(n)) &= l(M/IM) - h_p^0(\mathscr{F}) + \sum_{0<k\leqslant n} t_0^{(1)}(k)\, l(M/IM) \\ &\quad - \sum_{0<k\leqslant n} t_0^{(1)}(k)\, h_p^0(\mathscr{F}) - \sum_{0<k\leqslant n} t_0^{(1)}(k)\, h_p^1(\mathscr{F}) \\ &= \left(1 + \sum_{0<k\leqslant n} t_0^{(1)}(k)\right) l(M/IM) - \left(1 + \sum_{0<k\leqslant n} t_0^{(1)}(k)\right) h_p^0(\mathscr{F}) \\ &\quad - \sum_{0<k\leqslant n} t_0^{(1)}(k)\, h_p^1(\mathscr{F}) = t_0^{(2)}(n)\, l(M/IM) - t_0^{(2)}(n)\, h_p^0(\mathscr{F}) \\ &\quad - t_1^{(2)}(n)\, h_p^1(\mathscr{F}). \end{aligned}$$

In view of (5.1) and (5.2) this proves the case $d=2$.

So, let $d>2$. By induction we have for all $k>0$

$$h^0(\tilde{X}', \mathscr{H}'(k)) = t_0^{(d-1)}(k)\, l(M'/I'M') - \sum_{j=0}^{d-2} t_j^{(d-1)}(k)\, h_p^j(\mathscr{F}').$$

Applying (4.7) and observing that $l(M'/I'M') = l(M/IM)$, we may write

$$\begin{aligned} h^0(\tilde{X}', \mathscr{H}'(k)) &= t_0^{(d-1)}(k)\, l(M/IM) - t_0^{(d-1)}(k)\, h_p^0(\mathscr{F}) \\ &\quad - \sum_{j=1}^{d-2} (t_{j-1}^{(d-1)} + t_j^{(d-1)})(k)\, h_p^j(\mathscr{F}) \\ &\quad - t_{d-2}^{(d-1)}(k)\, h_p^{d-1}(\mathscr{F}). \end{aligned}$$

By (4.5) we have $h^0(\tilde{X}, \mathscr{H}(n)) = h^0(\tilde{X}, \mathscr{H}(0)) + \sum_{0<k\leqslant n} h^0(\tilde{X}', \mathscr{H}'(k))$ for all $n>0$. So, by (5.2) and the previous equality for $h^0(\tilde{X}', \mathscr{H}'(k))$ we get

$$\begin{aligned} h^0(\tilde{X}, \mathscr{H}(n)) = &\left(1 + \sum_{0<k\leqslant n} t_0^{(d-1)}(k)\right) l(M/IM) \\ &- \left(1 + \sum_{0<k\leqslant n} t_0^{(d-1)}(k)\right) h_p^0(\mathscr{F}) \\ &- \sum_{j=1}^{d-2} \sum_{0<k\leqslant n} (t_{j-1}^{(d-1)} + t_j^{(d-1)})(k)\, h_p^j(\mathscr{F}) \\ &- \sum_{0<k\leqslant n} t_{d-2}^{(d-1)}(k)\, h_p^{d-1}(\mathscr{F}) = t_0^{(d)}(n)\, l(M/IM) \\ &- \sum_{j=0}^{d-1} t_j^{(d)}(n)\, h_p^j(\mathscr{F}). \end{aligned}$$

This proves our claim. ∎

As an application of (5.5) we may calculate the *degree* of the exceptional sheaf $\tilde{\mathscr{F}} \mid E$. To do so, we prove the following auxiliary result:

(5.6) Lemma. *Let $d \geqslant 1$ and $j \in \{0, ..., d-1\}$. Then there is a polynomial $p_j^{(d)}$ of degree $<d-1$ such that*

$$t_j^{(d)}(n) = \binom{n+d-1}{d-1} \cdot \binom{d-1}{j} + p_j^{(d)}(n), \qquad \text{for all} \quad n \geqslant 1.$$

Proof (*Induction on d*). Let $d=1$. Then we must have $j=0$, and it suffices to choose $p_j^{(d)} = p_0^{(1)} = 0$.

So, let $d>1$. If $j=0$ it is sufficient to put $p_j^{(d)} = p_0^{(d)} = 0$ (cf. (5.4)). So let us assume $1 \leqslant j \leqslant d-2$. Then, by induction there are polynomials $p_{j-1}^{(d-1)}$ and $p_j^{(d-1)}$ of degree $<d-1$, and such that

$$t_{j-1}^{(d-1)}(k) = \binom{k+d-2}{d-2}\binom{d-2}{j-1} + p_{j-1}^{(d-1)}(k) \qquad (\forall k>0),$$

$$t_j^{(d-1)}(k) = \binom{k+d-2}{d-2}\binom{d-2}{j} + p_j^{(d-1)}(k) \qquad (\forall k>0).$$

As $\binom{d-2}{j-1} + \binom{d-2}{j} = \binom{d-1}{j}$ we may write for all $k>0$

$$(t_{j-1}^{(d-1)} + t_j^{(d-1)})(k) = \binom{k+d-2}{d-2}\binom{d-1}{j} + (p_{j-1}^{(d-1)} + p_j^{(d-1)})(k).$$

Thereby $p_{j-1}^{(d-1)}+p_j^{(d-1)}$ is a polynomial of degree $<d-2$. Define the function $\tilde{p}_j^{(d)}\colon \mathbb{N}\to\mathbb{N}$ by $\tilde{p}_j^{(d)}(n)=\sum_{0<k\leqslant n}(p_{j-1}^{(d-1)}+p_j^{(d-1)})(k)$. Then $\tilde{p}_j^{(d)}$ is represented by a polynomial of degree $<d-1$. As $\sum_{0<k\leqslant n}\binom{k+d-2}{d-2}=\binom{n+d-1}{d-1}-1$ we may put $p_j^{(d)}=p_j^{(d)}+\binom{d-1}{j}$ and apply (5.3)(ii)(b) to write $t_j^{(d)}$ in the requested form.

The case $j=d-1$ is treated similarly, using (5.3)(ii)(c) and observing that $\binom{d-1}{d-1}=1$. ▮

(5.7) Proposition. *Let $d>1$. Then the degree of the exceptional sheaf is given by*

$$\deg(\tilde{\mathscr{F}}\mid E)=l(M/IM)-\sum_{j=0}^{d-1}\binom{d-1}{j}h_p^j(\mathscr{F}).$$

Proof. By (5.5) and (5.6) we may write for $n>1$

$$h^0(\tilde{X},\mathscr{H}(n))=\binom{n+d-1}{d-1}\left[l(M/IM)-\sum_{j=0}^{d-1}\binom{d-1}{j}h_p^j(\mathscr{F})\right]$$
$$-\sum_{j=0}^{d-1}p_j^{(d)}(n)\,h_p^j(\mathscr{F}).$$

Thereby, the function $n\mapsto\sum_{j=0}^{d-1}p_j^{(d)}(n)\,h_p^j(\mathscr{F})$ is represented by a polynomial of degree $<d-2$. As $h^0(E,\tilde{\mathscr{F}}\mid E(n))=h^0(\tilde{X},\mathscr{H}(n))$, this proves our claim. ▮

(5.8) *Remark.* The degree of the exceptional sheaf is nothing else than the *multiplicity* of M at I. So, by (5.7) we have calculated this multiplicity, and thus get back a result of Trung [29].

The number

$$i_p(\mathscr{F}):=\sum_{j=0}^{d-1}\binom{d-1}{j}h_p^j(\mathscr{F})$$

plays an important role in this respect, as it measures how far away the multiplicity (or—equivalently—the degree of the exceptional sheaf) is away from what it should be in case $\mathscr{F}$ were a CM-sheaf.

(5.9) Corollary. *The cohomological Hilbert function $n\mapsto h^0(E,\tilde{\mathscr{F}}\mid E(n))$ of the exceptional sheaf $\tilde{\mathscr{F}}\mid E$ coincides with the corresponding Hilbert polynomial for all $n>-\min\{i>1\mid H_p^i(\mathscr{F})\neq 0\}$.*

Proof. The Hilbert polynomial of $\tilde{\mathscr{F}}\mid E$ is given by $n\mapsto\chi_{\tilde{\mathscr{F}}\mid E}(n)=\sum_i(-1)^i h^i(E,\tilde{\mathscr{F}}\mid E(n))=\sum_i(-1)^i h^i(\tilde{X},\mathscr{H}(n))$. Now, we may conclude the proof by (3.8) and (4.8). ▮

(5.10) EXAMPLE. Let $d=2$. Then

$$\min\{i>1 \mid H_p^i(\mathscr{F})\neq 0\}=2.$$

So $h^0(E, \tilde{\mathscr{F}} \mid E(n))=h^0(\tilde{X}, \mathscr{H}(n))$ should be presented by a (linear) polynomial for all $n\geqslant -1$. By (5.5) and observing that $t_0^{(2)}(n)=n+1$, $t_1^{(2)}(n)=n$ for $n\geqslant 1$ we get indeed $h^0(\tilde{X}, \mathscr{H}(n))=(n+1)\, l(M/IM)-(n+1)\, h_p^0(\mathscr{F})-nh_p^1(\mathscr{F})$ $(n\geqslant 1)$. We thus may write

$$h^0(E, \tilde{\mathscr{F}} \mid E(n))=[l(M/IM)-h_p^0(\mathscr{F})-h_p^1(\mathscr{F})]n-l(M/IM)-h_p^0(\mathscr{F})$$

for $n\geqslant -1$.

6. THE REES MODIFICATION

We keep the hypotheses and notations of the previous sections. The aim of this section is to study the cohomology of the blowing-up $\tilde{X}$ with coefficients in the Rees modification $\tilde{\mathscr{F}}$.

First, we describe the total module of sections

$$H_*^0(\tilde{X}, \tilde{\mathscr{F}})=\bigoplus_n H^0(\tilde{X}, \tilde{\mathscr{F}}(n))$$

in terms of the $\mathfrak{m}$-transform

$$D_{\mathfrak{m}}(M)=\Gamma(X-\{p\}, \mathscr{F})=: D_p(\mathscr{F}) \tag{6.1}$$

and of the modules $\bar{M}=M/H_{\mathfrak{m}}^0(M)$ and M.

We begin with a preliminary remark, recalling the canonical exact sequence [3]

$$0\to \bar{M}\to D_p(\mathscr{F})\to h_p^1(\mathscr{F})\to 0.$$

By means of this sequence we consider $\bar{M}$ as a submodule of $D_p(\mathscr{F})$. By the standard property of I we have $IH_p^1(\mathscr{F})=0$, thus $ID_p(\mathscr{F})\subseteq \bar{M}$. So for $x\in I$ we have a canonical multiplication map

$$x\colon D_p(\mathscr{F})\to \bar{M} \qquad (m\mapsto xm). \tag{6.2}$$

Finally, as $H_{\mathfrak{m}}^0(M)\cap IM=0$, we have canonical isomorphisms

$$I^nM \xrightarrow{\cong} I^n\bar{M} \qquad (\forall n>0). \tag{6.3}$$

Now, we may completely describe the graded $\mathfrak{R}(I)$-module $H_*(\tilde{X}, \tilde{\mathscr{F}})$:

(6.4) **Proposition.** *Let $d > 1$. Then we have:*

(i) $$H^0(\tilde{X}, \tilde{\mathscr{F}}(n)) = \begin{cases} D_p(\mathscr{F}), & \textit{for} \quad n < 0 \\ \bar{M}, & \textit{for} \quad n = 0 \\ I^n M = I^n \bar{M}, & \textit{for} \quad n > 0 \ (\textit{cf.}\ (6.3)). \end{cases}$$

(ii) *If $x \in I$, the multiplication map*

$$x^{(1)}: H^0(\tilde{X}, \tilde{\mathscr{F}}(n)) \to H^0(\tilde{X}, \tilde{\mathscr{F}}(n+1))$$

is given by the canonical map

$$\begin{cases} x: D_p(\mathscr{F}) \to D_p(\mathscr{F}), & \textit{for} \quad n < -1 \\ x: D_p(\mathscr{F}) \to \bar{M}, & \textit{for} \quad n = -1 \ (\textit{cf.}\ (6.2)) \\ x: I^n \bar{M} \to I^{n+1} \bar{M}, & \textit{for} \quad n \geqslant 0. \end{cases}$$

Proof. According to (6.3) the $\mathfrak{R}(I)$-modules $\mathfrak{R}(I, M)$ and $\mathfrak{R}(I, \bar{M})$ coincide in all positive degrees. So they both induce the same $\mathcal{O}_{\tilde{X}}$-sheaf $\tilde{\mathscr{F}}$. Therefore we may replace M by $\bar{M}$, thus assume that $H^0_{\mathfrak{m}}(M) = H^0_p(\mathscr{F}) = 0$, $\bar{M} = M$.

According to (3.1)(i) we may write

$$H^0_*(\tilde{X}, \tilde{\mathscr{F}}) = D_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M)).$$

So, it suffices to calculate the right-hand expression. To do so, we choose an element $y \in I$ which is a non-zero-divisor with respect to M. (This is possible as $H^0_I(M) = H^0_{\mathfrak{m}}(M) = 0$.) Then $y^{(1)} \in \mathfrak{R}(I)_{>0}$ is a non-zero-divisor with respect to $\mathfrak{R}(I, M)$.

Therefore we may write (cf. [3])

$$D_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M)) = \bigcup_{m \geqslant 0} (\mathfrak{R}(I, M) : \mathfrak{R}(I)^m_{>0})_{\mathfrak{R}(I, M)_{(y^{(1)})}}.$$

Thereby $\mathfrak{R}(I, M)_{(y^{(1)})}$ is a graded $\mathfrak{R}(I)$-module, whose nth homogeneous part is given by

$$(\mathfrak{R}(I, M)_{(y^{(1)})})_n = M_y.$$

Moreover, for $x \in I$, the multiplication map

$$x^{(1)}: (\mathfrak{R}(I, M)_{(y^{(1)})})_n \to (\mathfrak{R}(I, M)_{(y^{(1)})})_{n+1}$$

is given by the canonical map $x: M_y \to M_y$.

So, it remains to determine the R-modules

$$D_n := (D_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M)))_n = H^0(\tilde{X}, \tilde{\mathscr{F}}(n)) \qquad \text{for all} \quad n \in \mathbb{Z}.$$

Assume first that $n<0$. We have to show that D_n coincides with $D_p(\mathscr{F}) = D_I(M) = \bigcup_{m\geqslant 0}(M : I^m)_{M_\nu}$ [3].

To do so, let $u \in D_n$. Then, there is an $m \geqslant 0$ with $(\mathfrak{R}(I)_{>0})^m u \subseteq \mathfrak{R}(I, M)$. We thus may write $I^m u = \mathfrak{R}(I)_m u \subseteq \mathfrak{R}(I, M)_{m+n} = I^{n+m}M = M$, and have therefore $u \in D_p(\mathscr{F})$.

Now, conversely, let $v \in D_p(\mathscr{F}) \subseteq M_\nu = (\mathfrak{R}(I, M)_{(\nu^{(1)})})_n$. At the beginning of this section we have already seen that $ID_p(\mathscr{F}) \subseteq M$. Consequently we have $Iv \subseteq M$, thus $I^{-n}v \subseteq M$. This may be read as

$$(\mathfrak{R}(I)_{-n})v \subseteq \mathfrak{R}(I, M)_0 \subseteq \mathfrak{R}(I, M).$$

As $(\mathfrak{R}(I)_{>0})^{-n}$ is generated by $\mathfrak{R}(I)_{-n}$, we may write $(\mathfrak{R}(I)_{>0})^{-n} v \subseteq \mathfrak{R}(I, M)$. This show that $v \in D_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M))$, hence that $v \in D_n$.

Altogether this proves $H^0(\tilde{X}, \tilde{\mathscr{F}}(n)) = D_n = D_p(\mathscr{F})$ for all $n<0$.

So it remains to prove $D_n = I^n M$ for $n \geqslant 0$, or—equivalently—that the graded modules $\mathfrak{R}(I, M)$ and $D := D_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M))$ coincide in all degrees $n \geqslant 0$. In view of the canonical exact sequence

$$0 \to \mathfrak{R}(I, M) \to D \to H^1_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M)) \to 0,$$

this comes up to verify that the module $H := H^1_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M))$ vanishes in all degrees $\geqslant 0$. As $D = H^0_*(\tilde{X}, \tilde{\mathscr{F}})$ and as $\tilde{\mathscr{F}} = \mathfrak{R}(I, M)^\sim$ Serre's finiteness theorem [26] shows that D and $\mathfrak{R}(I, M)$ coincide in all large degrees $n \gg 0$. Therefore we have $H_n = 0$ for all $n \gg 0$.

Now, we consider the canonical sequences

$$0 \to I\mathfrak{R}(I, M) \to \mathfrak{R}(I, M) \to \mathrm{Gr}(I, M) \to 0$$

$$0 \to I\mathfrak{R}(I, M)(-1) \to \mathfrak{R}(I, M) \to M^{(0)} \to 0.$$

Applying local cohomology we obtain exact sequences

$$H^0_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) \to H^1_{\mathfrak{R}(I)_{>0}}(I\mathfrak{R}(I, M)) \to H \to H^1_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M))$$

$$H^0_{\mathfrak{R}(I)_{>0}}(M^{(0)}) \to H^1_{\mathfrak{R}(I)_{>0}}(I\mathfrak{R}(I, M))(-1) \to H \to H^1_{\mathfrak{R}(I)_{>0}}(M^{(0)}).$$

By (3.2) we already know that

$$H^0_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) = H^0_p(\mathscr{F})^{(0)} = 0, \qquad H^1_{\mathfrak{R}(I)_{>0}}(\mathrm{Gr}(I, M)) = H^1_p(\mathscr{F})^{(-1)}.$$

As $M^{(0)}$ is a $\mathfrak{R}(I)_{>0}$-torsion module we moreover have

$$H^0_{\mathfrak{R}(I)_{>0}}(M^{(0)}) = M^{(0)} \qquad \text{and} \qquad H^1_{\mathfrak{R}(I)_{>0}}(M^{(0)}) = 0.$$

So, the previous sequences give rise to isomorphisms $H_n \cong H_{n+1}$ for all $n \geqslant 0$. As $H_n = 0$ for $n \gg 0$ this completes our proof. ∎

Our next result is devoted to the total cohomology modules $H^i_*(\tilde{X}, \tilde{\mathscr{F}})$ for $0 < i < d-1$.

(6.5) PROPOSITION. *Let $d > 1$ and $0 < i < d-1$. Then*

$$H^i_*(\tilde{X}, \tilde{\mathscr{F}}) = \bigoplus_{j < -i} H^{i+1}_p(\mathscr{F})^{(j)}.$$

So, in particular

$$H^i(\tilde{X}, \tilde{\mathscr{F}}(n)) = \begin{cases} 0, & \text{for } n \geqslant -i \\ H^{i+1}_p(\mathscr{F}), & \text{for } n < -i, \end{cases}$$

and for $x \in I$ the multiplication maps

$$x^{(1)}\colon H^i(X, \tilde{\mathscr{F}}(n)) \to H^i(X, \mathscr{F}(n+1)) \qquad (n \in \mathbb{Z})$$

are zero maps.

Proof. Applying cohomology to the exact sequences (2.11) we get exact sequences

$$(*) \qquad H^{i-1}(\tilde{X}, \tilde{\mathscr{F}}(n)) \longrightarrow H^{i-1}(\tilde{X}, \mathscr{H}(n)) \longrightarrow H^i(\tilde{X}, \tilde{\mathscr{F}}(n+1)) \xrightarrow{e_n} H^i(\tilde{X}, \tilde{\mathscr{F}}(n)) \longrightarrow H^i(\tilde{X}, \mathscr{H}(n)) \longrightarrow H^{i+1}(\tilde{X}, \tilde{\mathscr{F}}(n+1))$$

for all $n \in \mathbb{Z}$ and all $i > 0$. By (3.8) and by (4.8), $H^i(\tilde{X}, \mathscr{H}(n))$ vanishes for all $n \geqslant -i$. So, for all $n \geqslant -i$ we get epimorphisms $e_n\colon H^i(\tilde{X}, \tilde{\mathscr{F}}(n+1)) \to H^i(\tilde{X}, \tilde{\mathscr{F}}(n))$.

As $H^i(\tilde{X}, \tilde{\mathscr{F}}(n)) = 0$ for $n \gg 0$, by Serre's finiteness theorem [26], we get

$$(**) \qquad H^i(\tilde{X}, \tilde{\mathscr{F}}(n)) = 0, \qquad \forall n \geqslant -i, \forall i > 1.$$

Applying $(**)$ with $i+1$ instead of i and with $n = -i$ we get $H^{i+1}(\tilde{X}, \tilde{\mathscr{F}}(-i)) = 0$. Applying $(**)$ with $n = -i$ we get $H^i(\tilde{X}, \tilde{\mathscr{F}}(-i)) = 0$. Finally, applying $(*)$ with $n = -i-1$ we thus get an isomorphism $H^i(\tilde{X}, \tilde{\mathscr{F}}(-i-1)) \cong H^i(\tilde{X}, \mathscr{H}(-i-1))$.

Observing (3.8) we therefore obtain $H^i(\tilde{X}, \tilde{\mathscr{F}}(-i-1)) \cong H^{i+1}_p(\mathscr{F})$, whenever $0 < i < d-1$.

Next, let $n < -i-1$. Then by (3.8)(i) and (ii) we have $H^{i-1}(\tilde{X}, \mathscr{H}(n)) = H^i(\tilde{X}, \mathscr{H}(n)) = 0$ and thus get an isomorphism $e_n\colon H^i(\tilde{X}, \tilde{\mathscr{F}}(n+1)) \to H^i(\tilde{X}, \tilde{\mathscr{F}}(n))$. So by descending induction on n we get

$$H^i(\tilde{X}, \tilde{\mathscr{F}}(n)) \cong H^{i+1}_p(\mathscr{F}), \qquad \forall n < -i \ (0 < i < d-1).$$

So, it remains to prove that $x^{(1)}$ acts trivially on $H^i_*(\tilde{X}, \tilde{\mathscr{F}})$ for all $x \in I$. To do so, observe that

$$e := \bigoplus_n e_n \colon H^i_*(\tilde{X}, \tilde{\mathscr{F}}(1)) \to H^i_*(\tilde{X}, \tilde{\mathscr{F}})$$

is a homomorphism of graded $\mathfrak{R}(I)$-modules. We already know that e_n is an isomorphism for $n < -i-1$. So we get the diagram

$$\begin{tikzcd} H^i(\tilde{X}, \tilde{\mathscr{F}}(n+1)) \arrow[r, "e_n", "\cong"'] \arrow[d, "x^{(1)}"] & H^i(\tilde{X}, \tilde{\mathscr{F}}(n)) \arrow[d, "x^{(1)}"] \\ H^i(\tilde{X}, \tilde{\mathscr{F}}(n+2)) \arrow[r, "e_{n+1}"] & H^i(\tilde{X}, \tilde{\mathscr{F}}(n+1)) \end{tikzcd} \qquad (n < -i-1).$$

For $n = -i-1$, $H^i(\tilde{X}, \tilde{\mathscr{F}}(n+2))$ vanishes, and so the left-hand vertical map vanishes. Now, we get the requested triviality of the maps $x^{(1)}$ from our diagrams by descending induction on n. ∎

(6.6) Complement. *Let $i \geq d$. Then*

$$H^i_*(\tilde{X}, \tilde{\mathscr{F}}) = 0.$$

Proof. We may write $H^i_*(\tilde{X}, \tilde{\mathscr{F}}) = H^{i+1}_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I, M))$ (cf. (3.1)(ii)). Writing $I = (x_1, ..., x_d)$ we have $\mathfrak{R}(I)_{>0} = (x_1^{(1)}, ..., x_d^{(1)})$. So $\mathfrak{R}(I)_{>0}$ is generated by at most d elements and therefore the functor $H^{i+1}_{\mathfrak{R}(I)_{>0}}$ vanishes (cf. [20]). ∎

Finally, we want to treat the module $H^{d-1}_*(\tilde{X}, \tilde{\mathscr{F}})$. Similar to the case of sheaf $\mathscr{H}$ we only determine the lengths $h^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n))$ of the R-modules $H^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n))$ for all $n \in \mathbb{Z}$. To do so, we consider the functions $s_j^{(d)}$ introduced in (4.9) and introduce their summatorial functions $\tilde{s}_j^{(d)}$ by

$$\text{(6.7)} \qquad \tilde{s}_j^{(d)}(n) = \sum_{k \leq n} s_j^{(d)}(k) \qquad (d \in \mathbb{N}; j = 0, ..., d-1).$$

Notice in particular, that by (4.10) we have

$$\text{(6.8)} \qquad \tilde{s}_0^{(d)}(n) = \binom{-n}{d} \qquad \text{for} \quad n \leq -d.$$

Now, using this notation we have

(6.9) Proposition. *Let $d > 1$. Then*

$$h^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n)) = \tilde{s}_0^{(d)}(n)\, l(M/IM) - \sum_{j=0}^{d-1} \tilde{s}_j^{(d)}(n)\, h_p^j(\mathscr{F}).$$

Proof. By the statement (**) in the proof of (6.5) we already know that $H^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n)) = 0$ for all $n \geqslant -d+1$. So, let $n < -d+1$. Then, by (3.8), we have $H^{d-2}(\tilde{X}, \tilde{\mathscr{H}}(n)) = 0$. So, applying the sequences (*) from the proof of (6.5) with $n < -d+1$ and $i = d-1$, we get exact sequences

$$0 \to H^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n+1)) \to H^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n)) \to H^{d-1}(\tilde{X}, \mathscr{H}(n)) \to 0.$$

As $H^{d-1}(\tilde{X}, \mathscr{H}(n)) = 0$ for $n \geqslant -d+1$ (cf. (4.8)), these sequences give $h^{d-1}(\tilde{X}, \tilde{\mathscr{F}}(n)) = \sum_{k \leqslant n} h^{d-1}(\tilde{X}, \mathscr{H}(k))$. By (4.11) and in view of (6.7) this proves our claim. ∎

(6.10) *Remarks.* (i) Proposition (6.5) is given by A Marca [1] in terms of local cohomology supported in $\mathfrak{R}(I)_{>0}$. The non-vanishing of the occurring modules is used to show that the *punctured affine cone* $\mathring{C} = \operatorname{Spec}(\mathfrak{R}(I)) - V(\mathfrak{R}(I)_{>0})$ over the blowing-up $\tilde{X} = \operatorname{Proj}(\mathfrak{R}(I))$ is not affine.

On the other hand we have for the ring of global sections over $\mathring{C}$

$$\Gamma(\mathring{C}, \mathcal{O}_{\mathring{C}}) = D_{\mathfrak{R}(I)_{>0}}(\mathfrak{R}(I)).$$

Now, consider the particular case, in which $\mathscr{F} = \mathcal{O}_X$, $H^0_p(\mathcal{O}_X) = 0$. Let $\mathfrak{R}\langle I \rangle$ be the *non-truncated Rees ring* $\cdots R \oplus R \oplus \cdots \oplus R \oplus I \oplus I^2 \oplus \cdots \cong R[IT, T^{-1}]$. Then $\mathfrak{R}(I)$, $\mathfrak{R}\langle I \rangle$, and $\Gamma(\mathring{C}, \mathcal{O}_{\mathring{C}})$ coincide in positive degrees and it holds moreover that $I\Gamma(\mathring{C}, \mathcal{O}_{\mathring{C}}) \subseteq \mathfrak{R}\langle I \rangle \subseteq \Gamma(\mathring{C}, \mathcal{O}_{\mathring{C}})$ (cf. (6.4)). So $\Gamma(\mathring{C}, \mathcal{O}_{\mathring{C}})$ is a finite integral extension of the noetherian graded ring $\mathfrak{R}\langle I \rangle$ hence noetherian. Corollary (3.8) also may be used to show that $\tilde{C} := \operatorname{Spec}(\mathfrak{R}(I)) - V(I\mathfrak{R}(I))$ is not affine. Here we have $\Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}}) \cong D_{\mathfrak{m}}(R)[T]$, so the ring $\Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}})$ is noetherian. Now, for arbitrary I, we may consider $C' = \operatorname{Spec}(\mathfrak{R}(I)) - Z$ where Z is the union of those irreducible components of $V(I\mathfrak{R}(I))$ that do not map onto $V(I)$. Then $\Gamma(C', \mathcal{O}_{C'})$ is the *symbolic Rees algebra of I* [3]. By Rees [22], this symbolic Rees algebra $\Gamma(C', \mathcal{O}_{C'})$ needs not be noetherian.

The symbolic Rees algebras of standard blowing-up have been studied in [5]. They play an important role for the construction of *Macaulayfications which preserve normality* and *regularity* [6].

(ii) One knows that $\pi\colon \tilde{X} \to X$ has a *Stein factorization*

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "\pi"] \arrow[dr, "\pi_0"'] & \arrow[d, phantom, "\circlearrowleft"] & X \\
& \operatorname{Spec}(\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})) \arrow[ur, "v"'] &
\end{tikzcd}$$

where v is induced by the canonical map

$$\pi^*\colon \Gamma(X, \mathcal{O}_X) \to \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}).$$

If $\mathscr{F} = \mathcal{O}_X$ and $H^0_p(\mathcal{O}_X) = 0$, (6.4) shows that π^* is given by $id\colon R \to R$. So, $id\colon X \to X$ is a Stein factor of π. This has been proved already in [7] and shows in particular the *connectedness of the exceptional fiber E*. This is not surprising: As we are blowing-up at an ideal I generated at p by a system of parameters, E is homeomorphic to a projective space of dimension $d-1$ over the function field $\kappa(p)$ of p.

Generally, blowing-up at an infinitesimal neighbourhood of a point does not lead to a connected exceptional fiber. This question is studied by Cottini [9].

7. B-Singular Bundles

In this section we consider a particular situation. First of all we assume that the affine scheme $X = \operatorname{Spec}(R)$ is regular. Moreover we assume that the sheaf $\mathscr{F}$ is locally free—thus a vector bundle—over $X - \{p\}$. So we may think of $\mathscr{F}$ as a bundle which has an isolated singularity at p. In this situation, $\mathscr{F}$ satisfies the finiteness condition (2.1) by Grothendieck's finiteness theorem [18].

But we do not content ourselves with this finiteness condition. We namely suppose that $\mathscr{F}$ has a singularity of *Buchsbaum type* at p, in short, that $\mathscr{F}$ is B-singular at p. By this we mean that the stalk $\mathscr{F}_p$ of $\mathscr{F}$ at p is a Buchsbaum module over the local ring $\mathcal{O}_{X,p}$. For this notion we refer to Stueckrad and Vogel [27]. One essential consequence of the fact that $\mathscr{F}_p$ is a Buchsbaum module is that any system $x_1, ..., x_d$ for which $\mathfrak{m}$ is a minimal prime divisor of $(x_1, ..., x_d)$ is a standard-system. As $R_{\mathfrak{m}} = \mathcal{O}_{X,p}$ is a regular local ring, this allows us to choose $x_1, ..., x_d \in \mathfrak{m}$ such that $(x_1, ..., x_d)_{\mathfrak{m}} = \mathfrak{m}R_{\mathfrak{m}}$, thus $(x_1, ..., x_d)\,\mathcal{O}_{X,p} = \mathfrak{m}_{X,p}$, hence it allows us to choose $I = \mathfrak{m}$. So the vanishing ideal $\mathfrak{m}$ of p is a standard ideal. We shall keep the hypothesis $I = \mathfrak{m}$ for this section:

$$(7.1) \qquad \begin{aligned} &\text{(i)} \quad (x_1, ..., x_d)\,\mathcal{O}_{X,p} = \mathfrak{m}_{X,p} \\ &\text{(ii)} \quad \tilde{X} = \operatorname{Proj}(\mathfrak{R}(\mathfrak{m})). \end{aligned}$$

Equation (7.1)(ii) means that the morphism

$$\pi\colon \tilde{X} = \operatorname{Proj}(\mathfrak{R}(\mathfrak{m})) \to X = \operatorname{Spec}(R)$$

is the blowing-up of X at the point p. As X is regular, this induces

$$(7.2) \qquad \tilde{X} \text{ is a regular scheme.}$$

Moreover, the exceptional fiber E is a projective space of dimension $d-1$ over the function field $\kappa(p)$ of p

$$(7.3) \qquad E = \mathbb{P}^{d-1}_{\kappa(p)}.$$

Now, let us consider the Rees modification $\tilde{\mathscr{F}}$. We know by [4], that $\tilde{\mathscr{F}}_q$ is a maximal Cohen–Macaulay module over $\mathcal{O}_{X,q}$ for any $q \in \tilde{X}$. As $\tilde{X}$ is regular, this means that $\tilde{\mathscr{F}}_q$ is free over $\mathcal{O}_{X,q}$. So $\tilde{\mathscr{F}}$ is locally free, hence a bundle, over $\tilde{X}$. Therefore in our particular situation we have resolved the singularity of $\mathscr{F}$ by blowing-up X at p and replacing $\mathscr{F}$ by $\tilde{\mathscr{F}}$.

Similarly we know that the stalk $\mathscr{E}_q$ of the exceptional sheaf $\mathscr{E} = \tilde{\mathscr{F}} \mid E$ is a maximal Cohen–Macaulay module over $\mathcal{O}_{E,q}$ for all $q \in E$. So by (7.3), $\mathscr{E}$ is a bundle over $E = \mathbb{P}^{d-1}_{\kappa(p)}$. We thus refer to $\mathscr{E}$ as the *exceptional bundle*.

Finally we assume that X is connected. Then the *rank* of the bundle $\mathscr{F} \mid X - \{p\}$ is defined. We denote this rank by r. Clearly the ranks of $\tilde{\mathscr{F}}$ and $\mathscr{E}$ equal r:

$$\text{(7.4)} \qquad \operatorname{rk}(\tilde{\mathscr{F}}) = \operatorname{rk}(\mathscr{E}) = r := \operatorname{rk}(\mathscr{F} \mid X - \{p\}).$$

We write $\mathscr{F}(p)$ for the $\kappa(p)$-vector space $\mathscr{F}_p \otimes \kappa(p) = M/\mathrm{m}M$. The dimension of this space then corresponds to the number $l(M/IM)$ which occurred in the earlier sections. $\dim(\mathscr{F}(p))$ also represents the minimal number of generators of the $\mathcal{O}_{X,p}$-module $\mathscr{F}_p$.

In what follows, the invariant

$$i_p(\mathscr{F}) = \sum_{j=0}^{d-1} \binom{d-1}{j} h_p^j(\mathscr{F})$$

introduced in (5.8) will play an important role. First of all we notice

(7.5) PROPOSITION. $r = \dim(\mathscr{F}(p)) - i_p(\mathscr{F})$.

Proof. As $\mathscr{E}$ is a bundle over $\mathbb{P}^{d-1}$, its degree and its rank coincide. Now, we conclude the proof by (5.7) and (7.5). ∎

(7.6) EXAMPLE. Let $d = 2$. Then $E = \mathbb{P}^1$. We claim

$$\mathscr{E} = \mathcal{O}_E(+1)^{h_p^1(\mathscr{F})} \oplus \mathcal{O}_E^{r - h_p^1(\mathscr{F})}.$$

Proof. As $\mathscr{E}$ splits into line bundles (cf. [25]) we may write

$$\mathscr{E} = \bigoplus_{i=1}^{s} \mathcal{O}_E(a_i)^{v_i}, \qquad \text{with} \quad a_1 > a_2 \cdots > a_s,\ v_1, ..., v_s \in \mathbb{N}, \sum v_i = r.$$

Then $h^0(E, \mathscr{E}(n)) = \sum_{i:\, n \geqslant -a_i} (n + a_i + 1)\, v_i$.

But now, by the shape of $h^0(E, \mathscr{E}(n))$ (cf. (5.10)), we may write $a_1 = -1$, $v_1 = h_p^1(\mathscr{F})$, $a_2 = 0$, $v_2 = r - h_p^1(\mathscr{F})$, thereby observing the relation $r = \dim(\mathscr{F}(p)) - h_p^0(\mathscr{F}) - h_p^1(\mathscr{F})$. ∎

Next we want to study the exceptional sheaf $\mathscr{E}$ in the case where $d > 2$. To do so, we begin with some preliminary results. For a moment we consider again the case where R is an arbitrary noetherian ring. We fix a standard system $x_1, ..., x_d \in \mathfrak{m}$ with respect to $\mathscr{F}$ at p and write I for the corresponding standard ideal. Now, generalizing (4.2) we put for $k \in \{1, ..., d\}$

(7.7) (i) $R^{[k]} := R/(x_1, ..., x_k)R;\ x_i^{[k]} = x_i/(x_1, ..., x_k)R$

$(i = k+1, ..., d)$

(ii) $I^{[k]} := IR^{[k]}$

(iii) $M^{[k]} := M/(x_1, ..., x_k)M$

(iv) $X^{[k]} := \mathrm{Spec}(R^{[k]}),\ \mathscr{F}^{[k]} := \mathscr{F} \mid X^{[k]} = (M^{[k]})^{\sim}$.

Clearly $X^{[k]}$ is a closed subscheme of X and satisfies $\dim_p(X^{[k]}) = d-k$. Moreover $x_{k+1}^{[k]}, ..., x_d^{[k]}$ is a standard system with respect to $\mathscr{F}^{[k]}$ at p, and $I^{[k]} \subseteq R^{[k]}$ is the corresponding standard ideal.

Now, (4.7) may be generalized to

(7.8) Lemma. *Let $j \in \{0, ..., d-k-1\}$. Then*

$$h_p^j(\mathscr{F}^{[k]}) = \sum_{l=0}^{k} \binom{k}{l} h_p^{j+l}(\mathscr{F}).$$

Proof. Without loss of generality we may assume that $(R, \mathfrak{m})$ is local. Now, we proceed by induction on k. The case $k=1$ is clear by (4.7). So, let $k>1$.

We may write $\mathscr{F}^{[k]} = (\mathscr{F}^{[k-1]})'$.

So (4.7) gives

$$h_p^j(\mathscr{F}^{[k]}) = h_p^j(\mathscr{F}^{[k-1]}) + h_p^{j+1}(\mathscr{F}^{[k-1]}).$$

By induction we may write

$$h_p^j(\mathscr{F}^{[k-1]}) = \sum_{l=0}^{k-1} \binom{k-1}{l} h_p^{j+l}(\mathscr{F})$$

and

$$h_p^{j+1}(\mathscr{F}^{[k-1]}) = \sum_{e=0}^{k-1} \binom{k-1}{e} h_p^{j+1+l}(\mathscr{F}).$$

So we get altogether

$$h_p^j(\mathscr{F}^{[k]}) = \binom{k-1}{0} h_p^j(\mathscr{F}) + \sum_{l=1}^{k-1} \left[\binom{k-1}{l-1} + \binom{k-1}{l}\right] h_p^{j+l}(\mathscr{F})$$
$$+ \binom{k-1}{k-1} h_p^{j+k}(\mathscr{F}) = \sum_{l=0}^{k} \binom{k}{l} h_p^{j+l}(\mathscr{F}). \quad \blacksquare$$

Now, we may prove:

(7.9) PROPOSITION. *Let $d \geqslant 2$, let $\kappa(p) = R/\mathfrak{m}$ be algebraically closed, and put*

$$m_p(\mathscr{F}) = \sum_{l=0}^{d-2} \binom{d-2}{l} h_p^{l+1}(\mathscr{F}).$$

Then the exceptional bundle $\mathscr{E}$ is uniform of splitting type

$$(\underbrace{+1, \ldots, +1}_{m_p(\mathscr{F})}, \underbrace{0, \ldots, 0}_{r - m_p(\mathscr{F})}).$$

Proof. We may assume that $(R, \mathfrak{m})$ is local. Now, let $L \subseteq E = \mathbb{P}^{d-1}$ be any projective line. We have to prove $\mathscr{E} \mid L = \mathcal{O}_L(+1)^{m_p(\mathscr{F})} \oplus \mathcal{O}_L^{r - m_p(\mathscr{F})}$ (cf. [25]).

For $d = 2$ we have $L = E$, and thus we may conclude the proof by (7.6). So, let $d > 2$. Choosing an appropriate regular system $x_1, \ldots, x_d$ of parameters of R we may assume that L already belongs to the exceptional fiber E' of the blowing-up $\pi' \colon \tilde{X}' \to X'$. Thereby $\cdot'$ is defined according to (4.2) and (4.3). Let $\mathscr{E}'$ be the exceptional bundle on E' which is related to the $\mathcal{O}_{X'}$-sheaf $\mathscr{F}'$. Then, by induction we have $\mathscr{E}' \mid L = \mathcal{O}_L(+1)^{m_p(\mathscr{F}')} \oplus \mathcal{O}_L^{r' - m_p(\mathscr{F}')}$, where $r' = \operatorname{rank}(\mathscr{F}' \mid X' - \{p\})$. Obviously $r' = r$. Moreover by (7.8), $m_p(\mathscr{F}') = h^1(\mathscr{F}'^{[d-3]} = \mathscr{F}^{[d-2]}) = m_p(\mathscr{F})$.

So, it remains to prove $\mathscr{E}' \mid L = \mathscr{E} \mid L$. It suffices to show $\mathscr{E}' = \mathscr{E} \mid E'$. To do so, we may assume $H_p^0(\mathscr{F}) = H_{\mathfrak{m}}^0(M) = 0$, as replacing M by $\bar{M} = M/H_{\mathfrak{m}}^0(M)$ does not affect $\tilde{\mathscr{F}}$ (and $\tilde{\mathscr{F}}'$). As the elements $x_1, \ldots, x_d$ form a regular sequence in the ring R, we have a canonical isomorphism $\operatorname{Gr}(\mathfrak{m}') \cong \operatorname{Gr}(\mathfrak{m})/x_1^{(1)} \operatorname{Gr}(\mathfrak{m})$, which allows us to write $E' = \operatorname{Proj}(\operatorname{Gr}(\mathfrak{m})/x_1^{(1)} \operatorname{Gr}(\mathfrak{m}))$.

So, $\mathscr{E} \mid E'$ is the sheaf induced by $\operatorname{Gr}(\mathfrak{m}, M)/x_1^{(1)} \operatorname{Gr}(\mathfrak{m}, M)$. As $H_p^0(\mathscr{F}) = 0$, we have the exact sequence

$$0 \longrightarrow \operatorname{Gr}(\mathfrak{m}, M) \xrightarrow{x_1^{(1)}} \operatorname{Gr}(\mathfrak{m}, M) \longrightarrow \operatorname{Gr}(\mathfrak{m}', M') \longrightarrow 0$$

(cf. [2]), which allows us to write $\operatorname{Gr}(\mathfrak{m}', M') = \operatorname{Gr}(\mathfrak{m}, M)/x_1^{(1)} \operatorname{Gr}(\mathfrak{m}, M)$. As $\mathscr{E}'$ is induced by $\operatorname{Gr}(\mathfrak{m}', M')$, this proves our claim.

(7.10) COROLLARY. *Keep the hypotheses of* (7.9) *and assume moreover that $\mathscr{F}$ is reflexive. Then the following conditions are equivalent*:

(i) *$\mathscr{F}_p$ is free over $\mathcal{O}_{X,p}$*

(ii) $i_p(\mathscr{F}) = 0$

(iii) $m_p(\mathscr{F}) = 0$

(iv) $H^i_*(\tilde{X}, \tilde{\mathscr{F}}) = 0$ *for* $1 \leqslant i < d-1$

(v) $H^i_*(E, \mathscr{E}) = 0$ *for* $1 \leqslant i < d-1$

(vi) $\mathscr{E}$ *is a direct sum of line bundles*

(vii) $\mathscr{E} = \mathcal{O}^r_E$.

Proof. As $\mathscr{F}$ is reflexive, $H^0_p(\mathscr{F}) = H^1_p(\mathscr{F}) = 0$. So (i) is equivalent to the vanishing of the modules $H^i_p(\mathscr{F})$ for all $i \in \{2, ..., d-1\}$, which indicates that $\mathscr{F}_p$ is CM over the regular ring $\mathcal{O}_{X,p}$. But this vanishing condition obviously is equivalent to each of the statements (ii), (iii), (iv), (v) (cf. (6.5), (3.8)). Statements and (vi) are equivalent by Horrocks' splitting criterion for bundles. (cf. [25]). That (vii) $\Rightarrow$ (vi) is trivial. Finally assume that $\mathscr{E}$ is a sum of line bundles. By what we know already, we get $m_p(\mathscr{F}) = 0$. Now, (7.9) induces $\mathscr{E} = \mathcal{O}^r_E$. This proves (vi) $\Rightarrow$ (vii).

(7.11) Example. Let $(R, \mathfrak{m})$ be regular and local of dimension $d > 2$. Let M_i be the ith syzygy of $\mathfrak{m}$ and put $\mathscr{F}_i = \tilde{M}_i$ $(i = 1, ..., d-2)$. Then $\mathscr{F}_i$ is a reflexive sheaf and moreover locally free outside of the closed point p of X. We put

$$r_i = \operatorname{rank}(\mathscr{F}_i) = \operatorname{rank}(M_i); \qquad e_i = \dim(\mathscr{F}_i(p)) = \dim_{R/\mathfrak{m}}(M_i/\mathfrak{m}M_i).$$

Then there are short exact sequences

$$0 \to M_1 \to R^d \to \mathfrak{m} \to 0$$

$$0 \to M_i \to R^{e_{i-1}} \to M_{i-1} \to 0 \qquad (1 < i < d-1),$$

which show that

(i) $$r_i = d-1; \qquad r_i = e_{i-1} - r_{i-1} \ (1 < i < d-1).$$

Moreover, applying local cohomology to the above sequences and observing that $H^j_{\mathfrak{m}}(R^s) = 0$, $\forall j < d$, we get

$$H^j_{\mathfrak{m}}(M_1) = H^{j-1}_{\mathfrak{m}}(\mathfrak{m}); \qquad H^j_{\mathfrak{m}}(M_i) = H^{j-1}_{\mathfrak{m}}(M_{i-1}) \ (0 < j < d, 1 < i < d-1).$$

Applying cohomology to the exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$$

we get

$$H^{j-1}_{\mathfrak{m}}(\mathfrak{m}) = \begin{cases} 0, & j \neq 2, d+1 \\ R/\mathfrak{m}, & j = 2, \end{cases}$$

thus

$$H^j_{\mathfrak{m}}(M_i) = \begin{cases} 0, & j \neq i+1, \\ R/\mathfrak{m}, & j = i+1 \end{cases} \quad (0 < j < d, 1 < i < d-1).$$

This shows in particular that the modules M_i are of Buchsbaum type [27] and gives

$$h_p^j(\mathscr{F}_i)=\begin{cases}0, & j\neq i+1\\ 1, & j=i+1\end{cases}\quad (0<j<d,\ 1<i<d-1).$$

So we obtain

$$\text{(ii)}\qquad i_p(\mathscr{F}_i)=\binom{d-1}{i+1},\qquad m_p(\mathscr{F}_i)=\binom{d-2}{i}.$$

By (7.5) we get $e_i=r_i+i_p(\mathscr{F}_i)=r_i+\binom{d-1}{i+1}$. If $i>1$, we thus may write $e_{i-1}=r_{i-1}+\binom{d-1}{i}$. So (i) leads to $e_i=r_i+\binom{d-1}{i+1}=e_{i-1}-r_{i-1}+\binom{d-1}{i+1}=\binom{d-1}{i+1}+\binom{d-1}{i}=\binom{d}{i+1}$. Consequently $r_i=e_i-\binom{d-1}{i+1}=\binom{d-1}{i}$. Thus

$$\text{(iii)}\quad r_i=\operatorname{rank}(\mathscr{F}_i)=\binom{d-1}{i},\qquad e_i=\dim(\mathscr{F}_i(p))=\binom{d}{i+1}.$$

So the exceptional bundle $\mathscr{E}_i$ induced by $\mathscr{F}_i$ is of rank $\binom{d-1}{i}$ and of splitting type

$$\text{(iv)}\qquad (\underbrace{+1,\dots,+1}_{\binom{d-2}{i}},\underbrace{0,\dots,0}_{\binom{d-2}{i-1}}).$$

Clearly the values of r_i and e_i also may be obtained by using a *Koszul resolution* of $\mathfrak{m}$.

(7.12) *Remark.* Let $d>2$, depth $(\mathscr{F}_p)=d-1$. Then $h_p^i(\mathscr{F})=0$ for $i<d-1$ induces

$$\text{(i)}\qquad i_p(\mathscr{F})=m_p(\mathscr{F})=h_p^{d-1}(\mathscr{F})>0.$$

Putting $r=\operatorname{rank}(\mathscr{F}_p)$ we thus get by (7.10)

$$\text{(ii)}\qquad 0<h_p^{h-1}(\mathscr{F})\leqslant r.$$

Writing $e=\dim_{\kappa(p)}(\mathscr{F}(p))$, observing the additivity of the rank, and making use of the formula of Auslander and Buchsbaum we get the following minimal free resolution of the stalk $\mathscr{F}_p$:

$$0\longrightarrow \mathcal{O}_{X,p}^{e-r}\xrightarrow{\ \varepsilon\ }\mathcal{O}_{X,p}^{e}\longrightarrow \mathscr{F}_p\longrightarrow 0.$$

Thereby $\operatorname{im}(\varepsilon)$ is generated by $e-r$ elements $(x_{1,j},\dots,x_{e,j})\in\mathcal{O}_{X,p}^e$ $(j=1,\dots,e-r)$. As our resolution is minimal, we have $x_{i,j}\in\mathfrak{m}_{X,p}$ for all possible choices of i and j. Now, let $\mathfrak{p}\in\operatorname{Spec}(\mathcal{O}_{X,p})-\{\mathfrak{m}_{X,p}\}$. Then $(\mathscr{F}_p)_\mathfrak{p}$ is free over $(\mathcal{O}_{X,p})_\mathfrak{p}$ and so the previous sequence splits after localizing at $\mathfrak{p}$.

Therefore the elements $(x_{1,j}, ..., x_{e,j})$ form a part of a free basis of $(\mathcal{O}_{X,p})^e_{\mathfrak{p}}$ and thus satisfy $\sum_i (\mathcal{O}_{X,p})_{\mathfrak{p}}\, x_{i,j} = (\mathcal{O}_{X,p})^e_{\mathfrak{p}}$, hence $\sum_i \mathcal{O}_{X,p} x_{i,j} \nsubseteq \mathfrak{p}$. Therefore the ideal $\sum_i \mathcal{O}_{X,p} x_{i,j}$ is $\mathfrak{m}_{X,p}$-primary and thus has at least d generators, it follows that

$$\text{(iii)} \qquad h_p^{d-1}(\mathcal{F}) + r = e \geqslant d.$$

So, combining (ii) and (iii) we get

$$\text{(iv)} \qquad \operatorname{depth}(\mathcal{F}_p) = d-1 \Rightarrow 2r \geqslant d \qquad (d > 2).$$

The "least singular case" is the one with $h_p^{d-1}(\mathcal{F}) = 1$. Choosing $(R, \mathfrak{m})$ regular local, this case is realized in setting $\mathcal{F} = (R^d/(x_1, ..., x_d)R)^\sim$, where $x_1, ..., x_d$ is a regular system of parameters of R. This case exactly corresponds to the sheaves $\mathcal{F}_{d-2}$ of the previous example:

$$\text{(v)} \qquad \text{If } \operatorname{depth}(\mathcal{F}_p) = d-1, \text{ then}$$
$$h_p^{d-1}(\mathcal{F}) = 1 \Leftrightarrow \mathcal{F} \cong \mathcal{F}_{d-2}.$$

8. Rees Modification versus Pull-Back

Let $X = \operatorname{Spec}(R)$ be an arbitrary affine noetherian scheme, let $I \subseteq R$ be an arbitrary ideal, and let

$$\pi\colon \operatorname{Proj}(\mathfrak{R}(I)) = \tilde{X} \to X = \operatorname{Spec}(R)$$

be the blowing-up of X at $\operatorname{Spec}(R/I)$. Let $\mathcal{F}$ be a coherent sheaf induced by a finitely generated R-module M. The aim of this section is to compare the Rees modification $\tilde{\mathcal{F}}$ of $\mathcal{F}$ (hence the $\mathcal{O}_{\tilde{X}}$-sheaf induced by the Rees module $\mathfrak{R}(I, M)$) with the pull-back $\pi^*\mathcal{F}$ of $\mathcal{F}$, hence the *inverse image* sheaf of $\mathcal{F}$ with respect to π. Note that this pull-back is induced by the tensor product of M with the Rees ring $\mathfrak{R}(I)$ [19]:

$$\text{(8.1)} \qquad \pi^*\mathcal{F} = (M \otimes_R \mathfrak{R}(I))^\sim.$$

We put $\bar{M} = M/\Gamma_I(M)$, $\bar{\mathcal{F}} = \bar{M}^\sim$, where $\Gamma_I(M)$ is the I-torsion of M (this notation is compatible with our earlier use of the symbol $\bar{\ }$). By $\tilde{\bar{\mathcal{F}}}$ we denote the Rees modification of $\bar{\mathcal{F}}$ with respect to π. First we need, that $\tilde{\mathcal{F}}$ and $\tilde{\bar{\mathcal{F}}}$ coincide:

(8.2) Lemma. *The canonical map* $h\colon \mathfrak{R}(I, M) \to \mathfrak{R}(I, \bar{M})$ *induces an isomorphism* $\tilde{h}\colon \tilde{\mathcal{F}} \to \tilde{\bar{\mathcal{F}}}$.

Proof. In degree n, h is given by the canonical projection $h_n: I^nM \to I^n\bar{M}$. So it remains to verify that the kernel $I^nM \cap \Gamma_I(M)$ of h_n vanishes for all $n \gg 0$. To do so, we remark that $I^r\Gamma_I(M) = 0$ for some $r \in \mathbb{N}_0$. Moreover, applying Artin–Rees to the pair $\Gamma_I(M) \subseteq M$, we find an $s \in \mathbb{N}_0$ such that $I^nM \cap \Gamma_I(M) \subseteq I^{n-s}\Gamma_I(M)$ for all $n \geqslant s$. So, for $n \geqslant r+s$ the module $I^nM \cap \Gamma_I(M)$ vanishes.

(8.3) LEMMA. *The kernel of the canonical surjection* $m: \mathfrak{R}(I) \otimes_R M \to \mathfrak{R}(I, M)$ *is an I-torsion module*

$$\ker(m) \subseteq \Gamma_I(\mathfrak{R}(I) \otimes_R M).$$

Proof. m is given in degree n by the multiplication map $m_n: I^n \otimes M \to I^nM$ defined by $a \otimes x \mapsto ax$. It suffices to prove the inclusions

$$\ker(m_n) \subseteq \Gamma_I(I^n \otimes M) \qquad (n \in \mathbb{N}).$$

To do so, we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^R(R/I^n, M) \arrow[r, "\delta"] & I^n \otimes M \arrow[r] \arrow[d, "m_n"] & R \otimes M \arrow[r] \arrow[d, "\cong"', "m_0"] & R/I^n \otimes M \arrow[r] \arrow[d, "\cong"] & 0 \\
 & 0 \arrow[r] & I^nM \arrow[r] & M \arrow[r] & M/I^nM \arrow[r] & 0,
\end{tikzcd}$$

which shows that $\ker(m_n) \subseteq \delta(\mathrm{Tor}_1^R(R/I^n, M))$, hence that $I^n \ker(m_n) = 0$. Consequently $\ker(m_n) \subseteq \Gamma_I(I^n \otimes M)$. ∎

Now, we may compare the pull-back $\pi^*\mathscr{F}$ to the Rees modification $\tilde{\mathscr{F}}$. To do so, we introduce the sheaf of ideals

$$\mathscr{J} := I\mathfrak{R}(I)^{\sim} \subseteq \mathcal{O}_{\tilde{X}}. \tag{8.4}$$

$\mathscr{J}$ is the *exceptional divisor* of the blowing-up π, and may be viewed as the kernel of the canonical map $\mathcal{O}_{\tilde{X}} \to i_*\mathcal{O}_E$, where $i: E \to \tilde{X}$ stands for the inclusion map from the exceptional fiber $E = \mathrm{Proj}(\mathrm{Gr}(I))$ to $\tilde{X}$.

If $\mathscr{G}$ is a quasi-coherent $\mathcal{O}_{\tilde{X}}$-sheaf, we denote by $\Gamma_{\mathscr{J}}(\mathscr{G})$ the $\mathscr{J}$-torsion subsheaf of $\mathscr{G}$. One verifies easily, that for a graded $\mathfrak{R}(I)$-module N, the following holds true:

$$\Gamma_{\mathscr{J}}(\tilde{N}) = \Gamma_I(N)^{\sim}. \tag{8.5}$$

Now, we may state the anounced comparison result:

(8.6) PROPOSITION. *There is a canonical short exact sequence*

$$0 \longrightarrow \Gamma_{\mathscr{J}}(\pi^*\mathscr{F}) \longrightarrow \pi^*\mathscr{F} \xrightarrow{\tilde{m}} \tilde{\mathscr{F}} \longrightarrow 0.$$

Proof. The exact sequence

$$0 \longrightarrow \ker(m) \longrightarrow \mathfrak{R}(I) \otimes M \xrightarrow{m} \mathfrak{R}(I, M) \longrightarrow 0$$

together with $\ker(m) \subseteq \Gamma_I(\mathfrak{R}(I) \otimes M)$ leads to a short exact sequence

$$0 \longrightarrow \ker(\tilde{m}) \longrightarrow \pi^*\mathscr{F} \xrightarrow{\tilde{m}} \tilde{\mathscr{F}} \longrightarrow 0,$$

with $\ker(\tilde{m}) \subseteq \Gamma_{\mathscr{I}}(\pi^*\mathscr{F})$ (cf. (8.1), (8.5)). To calculate $\tilde{\mathscr{F}}$ we may replace M by $\bar{M}$ (cf. (8.2)), thus assume $\Gamma_I(M) = 0$. Consequently we get $\Gamma_I(\mathfrak{R}(I, M)) = 0$, thus $\Gamma_{\mathscr{I}}(\tilde{\mathscr{F}}) = 0$ (cf. (8.5)). As $\pi^*\mathscr{F}/\ker(\tilde{m}) \subseteq \tilde{\mathscr{F}}$ it follows that $\ker(\tilde{m}) = \Gamma_{\mathscr{I}}(\pi^*\mathscr{F})$. ∎

The following result shows that the Rees modification and the pull-back of a B-singular sheaf do not coincide.

(8.7) Proposition. *Let $X = \operatorname{Spec}(R)$ be regular and connected. Let $p \in X$ be a closed point and let $\pi: \tilde{X} \to X$ be the blowing-up of X at p. Let $\mathscr{F}$ be an $\mathcal{O}_X$-bundle with a unique singularity at p, and assume that this singularity is of Buchsbaum type. Let E be the exceptional fiber of π and let $\mathscr{E} = \tilde{\mathscr{F}} \mid E$ be the exceptional bundle. Finally let $\mathscr{K}$ be the kernel of the canonical epimorphism $\tilde{m} \mid E: \pi^*\mathscr{F} \mid E \to \mathscr{E}$.*

Then $\mathscr{K}$ is a bundle whose rank is given by

$$\operatorname{rk}(\mathscr{K}) = \dim_{\kappa(p)}(\mathscr{F}(p)) - \operatorname{rk}(\mathscr{F} \mid X - \{p\}) = i_p(\mathscr{F}).$$

Proof. Let $\mathfrak{m}$ be the maximal ideal which corresponds to p. Then $\pi^*\mathscr{F} \mid E$ is the $\mathcal{O}_E$-sheaf induced by the graded $\operatorname{Gr}(\mathfrak{m})$-module $\operatorname{Gr}(\mathfrak{m}) \otimes_R M = \operatorname{Gr}(\mathfrak{m}) \otimes M/\mathfrak{m}M$. Moreover $\tilde{m} \mid E$ is induced by the canonical homomorphism $\operatorname{Gr}(\mathfrak{m}) \otimes_R M \to \operatorname{Gr}(\mathfrak{m}, M)$. As $R_\mathfrak{m}$ is a regular local ring of dimension d, $E = \operatorname{Proj}(\operatorname{Gr}(\mathfrak{m}))$ equals the projective space $\mathbb{P}^{d-1}$ over $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} = \kappa(p)$. So we have $\pi^*\mathscr{F} \mid E \cong \mathcal{O}_{\mathbb{P}^{d-1}}^{\dim(M/\mathfrak{m}M)} = \mathcal{O}_{\mathbb{P}^{d-1}}^{\dim(\mathscr{F}(p))}$. We therefore get an exact sequence of bundles

$$0 \to \mathscr{K} \to \mathcal{O}_{\mathbb{P}^{d-1}}^{\dim(\mathscr{F}(p))} \to \mathscr{E} \to 0.$$

Observing the additivity of rank and degree we get our claim by (5.7) and (7.5). ∎

(8.8) Corollary. *Let X be as in* (8.7). *Assume that* $\dim(X) > 2$, *and let $\mathscr{F}$ be a reflexive $\mathcal{O}_X$-sheaf which is locally free outside the closed point p. Assume that $\mathscr{F}_p$ is a B-module. Then the following are equivalent*:

(i) *$\mathscr{F}_p$ is a free $\mathcal{O}_{X,p}$-module.*

(ii) *The canonical map $\pi^*\mathscr{F} \xrightarrow{\tilde{m}} \tilde{\mathscr{F}}$ is an isomorphism.*

(iii) *$\pi^*\mathscr{F}$ has no $\mathscr{I}$-torsion.*

Proof. Parts (ii) and (iii) are equivalent by (8.6). That (ii) $\Rightarrow$ (i) follows from (8.7) and (7.10). To prove (i) $\Rightarrow$ (iii) use (7.10) and the well known fact that the pull-back of a locally free sheaf is again locally free [19]. ∎

(8.9) *Remark.* In the proof of (8.7) we have seen that in the B-singular case we may write

$$\text{(i)} \qquad \pi^*\mathscr{F} \mid E = \mathcal{O}_E^{\dim(\mathscr{F}(p))},$$

where $E = \mathbb{P}^{d-1}_{\kappa(p)}$. In the notations of (8.7), this leads us to the exact sequence

$$\text{(ii)} \qquad 0 \to \mathscr{K} \to \mathcal{O}_{\mathbb{P}^{d-1}}^{\dim(\mathscr{F}(p))} \to \mathscr{E} \to 0.$$

As the modules $H^i_*(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}})$ are well known (cf. [26]), we may apply cohomology to the above sequence and then use (3.8), (4.11), and (5.5) to determine the modules

$$H^i_*(E, \mathscr{K}) \qquad \text{for} \quad i \neq d-1,$$

and the cohomological Hilbert functions

$$n \mapsto h^{d-1}(E, \mathscr{K}(n)), h^0(E, \mathscr{K}(n)).$$

This is left to the reader.

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